FIRST MIDTERM
MATH 104B, UCSD, WINTER 18

You have 80 minutes.

There are 4 problems, and the total number of points is 70. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________________

Signature:________________________________

Student ID #:________________________________

Problem	Points	Score
1	15	
2	15	
3	30	
4	10	
5	10	
6	10	
Total	70	

1. (15pts) (i) *Give the definition of* $\pi(x)$.

The number of primes up to x.

(ii) *Give the definition of the Möbius function.*

The function $\mu\colon \mathbb{N} \longrightarrow \mathbb{N}$ defined by the rule

$$\mu(n) = \begin{cases} (-1)^\nu & \text{if } n \text{ is the product of } \nu \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *Give the definition of the fractional part.*

If x is real number and $\llcorner x \lrcorner$ is the largest integer less than x then the fractional part is

$$\{\, x \,\} = x - \llcorner x \lrcorner$$

2. (15pts) *Let x, x_1 and x_2 be real numbers and let n be an integer. Prove that*

(i) $\llcorner x+n \lrcorner = \llcorner x \lrcorner + n$.

As $\llcorner x \lrcorner \leq x$ it follows that $\llcorner x \lrcorner + n \leq \llcorner x+n \lrcorner$. As $\llcorner x+n \lrcorner \leq x+n$ it follows that $\llcorner x+n \lrcorner - n \leq x$. As the LHS is an integer it follows that $\llcorner x+n \lrcorner - n \leq \llcorner x \lrcorner$. Adding n to both sides we get $\llcorner x+n \lrcorner \leq \llcorner x \lrcorner + n$. As we have an inequality both ways we must have an equality $\llcorner x+n \lrcorner = \llcorner x \lrcorner + n$..

(ii) $\llcorner x_1 \lrcorner + \llcorner x_2 \lrcorner \leq \llcorner x_1 + x_2 \lrcorner$.

Since

$$\llcorner x_i \lrcorner \leq x_i \qquad \text{for } i = 1,2 \text{ we have} \qquad \llcorner x_1 \lrcorner + \llcorner x_2 \lrcorner \leq x_1 + x_2.$$

As the LHS is an integer it follows that $\llcorner x_1 \lrcorner + \llcorner x_2 \lrcorner \leq \llcorner x_1 + x_2 \lrcorner$.

(iii) *Assuming that n is a natural number, prove that*

$$\llcorner \frac{x}{n} \lrcorner = \llcorner \frac{\llcorner x \lrcorner}{n} \lrcorner.$$

As

$$\llcorner x \lrcorner \leq x \qquad \text{it follows that} \qquad \frac{\llcorner x \lrcorner}{n} \leq \frac{x}{n}.$$

But then

$$\llcorner \frac{\llcorner x \lrcorner}{n} \lrcorner \leq \frac{x}{n} \qquad \text{so that} \qquad \llcorner \frac{\llcorner x \lrcorner}{n} \lrcorner \leq \llcorner \frac{x}{n} \lrcorner,$$

as the LHS is an integer.

On the other hand, as

$$\llcorner \frac{x}{n} \lrcorner \leq \frac{x}{n} \qquad \text{it follows that} \qquad n \llcorner \frac{x}{n} \lrcorner \leq x.$$

so that

$$\llcorner \frac{x}{n} \lrcorner \leq \frac{\llcorner x \lrcorner}{n} \qquad \text{and so} \qquad \llcorner \frac{x}{n} \lrcorner \leq \llcorner \frac{\llcorner x \lrcorner}{n} \lrcorner.$$

As we have an inequality both ways, we have equality.

3. (30pts) *Let $f\colon \mathbb{N} \longrightarrow \mathbb{C}$ be a function and define $F\colon \mathbb{N} \longrightarrow \mathbb{C}$ by the rule*

$$F(n) = \sum_{d|n} f(d).$$

(i) *Show that if f is multiplicative then F is multiplicative.*

Suppose that m and n are coprime. Note that if d divides mn then $d = d_1 d_2$ where d_1 divides m and d_2 divides n. We have

$$\begin{aligned} F(mn) &= \sum_{d|mn} f(d) \\ &= \sum_{d_1|m, d_2|n} f(d_1 d_2) \\ &= \sum_{d_1|m} \sum d_2|n f(d_1) f(d_2) \\ &= \sum_{d_1|m} f(d_1) \sum_{d_2|n} f(d_2) \\ &= F(m)F(n). \end{aligned}$$

Thus F is multiplicative.

(ii) *If the function*

$$M\colon \mathbb{N} \longrightarrow \mathbb{Z} \qquad \text{is defined by} \qquad M(n) = \sum_{d|n} \mu(d)$$

then show that

$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider

$$M(n) = \sum_{d|n} \mu(d).$$

As $\mu(n)$ is multiplicative both sides are multiplicative. If $n = p^e$ is a power of a prime then

$$M(p^e) = \mu(1) + \mu(p) + \mu(p^2) + \cdots = 1 - 1 + 0 + \cdots + 0 = 0.$$

Thus $M(n) = 0$ unless $n = 1$ in which case $M(1) = 1$.

(iii) *Show that*

$$f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d}).$$

We have

$$\begin{aligned}
\sum_{d|n} \mu(d) F(\frac{n}{d}) &= \sum_{d_1 d_2 = n} \mu(d_1) F(d_2) \\
&= \sum_{d_1 d_2 = n} \mu(d_1) \sum_{d|d_2} f(d) \\
&= \sum_{d_1 d|n} \mu(d_1) f(d) \\
&= \sum_{d|n} f(d) \sum_{d_1|n/d} \mu(d_1) \\
&= \sum_{d|n} f(d) M(n/d) \\
&= f(n).
\end{aligned}$$

4. (10pts) *Show that*

$$\pi(n) \geq \frac{\log n}{2 \log 2}.$$

Let $r = \pi(n)$ and let $p_1, p_2, \dots, p_r$ be the first r primes, so that $p_1, p_2, \dots, p_r$ are the primes up to n. Note that we may form 2^r distinct square-free natural numbers m which are only divisible by $p_1, p_2, \dots, p_r$. For each prime p_i we either choose m coprime to p_i or divisible by p_i. On other hand there are at most $\sqrt{n}$ perfect squares up n.
Now any natural number l is the product of a perfect square and a square-free number m. If $l \leq n$ then $m \leq n$ and so m is divisible by $p_1, p_2, \dots, p_r$. Thus there are at most $2^r\sqrt{n}$ numbers up to n. On the other hand there are n natural numbers up to n.
Thus

$$2^{\pi(n)}\sqrt{n} \geq n \qquad \text{so that} \qquad 2^{\pi(n)} \geq \sqrt{n}.$$

Taking logs we see that

$$\begin{aligned} \pi(n) \log 2 &= \log 2^{\pi(n)} \\ &\geq \log \sqrt{n} \\ &= \frac{1}{2} \log n. \end{aligned}$$

Thus

$$\pi(n) \geq \frac{\log n}{2 \log 2}.$$

5. Show that

$$\sum_{i<j=1}^{\infty} \llcorner \frac{x}{p_i p_j} \lrcorner = x \sum_{p_i p_j \le x, p_i < p_j} \frac{1}{p_i p_j} + O(x).$$

Note that

$$\begin{aligned}\sum_{i,j=1:p_i<p_j}^{\infty} \llcorner \frac{x}{p_i p_j} \lrcorner &= \sum_{p_i p_j \le x, p_i < p_j} \llcorner \frac{x}{p_i p_j} \lrcorner \\ &\le \sum_{p_i p_j \le x, p_i < p_j} \frac{x}{p_i p_j} - E,\end{aligned}$$

where

$$E \le \sum_{p_i p_j \le x, p_i < p_j} 1.$$

The term on the RHS is the number of ways to pick two primes p_i, p_j such that $p_i < p_j$ and $p_i p_j \le x$. Let $y = p_i p_j$. Then y determines p_i and p_j by unique factorisation and y is a natural number between 1 and x so that $y \le \llcorner x \lrcorner \le x$.
Thus E is at most x and so $-E = O(1)$.

Bonus Challenge Problems

6. (10pts) *Show that there is a constant* γ *such that*

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right).$$

Let

$$\alpha_k = \log k - \log(k-1) - \frac{1}{k} \qquad \text{and} \qquad \gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

Note that

$$1 - \gamma_n = \sum_{k=2}^{n} \alpha_k.$$

Note also that

$$\begin{aligned} \int_{k-1}^{k} \frac{1}{x}\, \mathrm{d}x &= [\log x]_{k-1}^{k} \\ &= \log k - \log(k-1) \end{aligned}$$

is the area under the curve $y = 1/x$ over the interval $k - 1 \leq x \leq k$. On the other hand $1/k$ is the area over the interval $k - 1 \leq x \leq k$ inside the largest rectangle inscribed between the x-axis and the curve $y = 1/x$.

It follows that α_k is the difference between these two areas, so that α_k is positive. Note that if we drop these areas down to the region between $x = 0$ and $x = 1$ then all of these areas fit into the unit square bounded by $y = 0$ and $y = 1$.

Thus $0 < 1 - \gamma_n < 1$ is bounded and monotonic increasing. It follows that $1 - \gamma_n$ tends to a limit. Define γ by the formula:

$$\lim_{n\to\infty} (1 - \gamma_n) = 1 - \gamma.$$

Finally note that the difference

$$\begin{aligned} \gamma_n - \gamma &= (1 - \gamma) - (1 - \gamma_n) \\ &= \sum_{k=n+1}^{\infty} \alpha_k \end{aligned}$$

is represented by an area which fits inside a box with one side 1 and the other side $1/n$, so that it is less than $1/n$. Thus

$$\gamma_n - \gamma = O\left(\frac{1}{n}\right).$$

7. (10pts) *Show that*

$$\pi(x) = O\left(\frac{x}{\log\log x}\right).$$

The number of integers up to x not divisible by the first r primes $p_1, p_2, \ldots, p_r$ is

$$A(x,r) = \llcorner x \lrcorner - \sum_{i=1}^{r} \llcorner \frac{x}{p_i} \lrcorner + \sum_{i \neq j \leq r} \llcorner \frac{x}{p_i p_j} \lrcorner + \cdots + (-1)^r \llcorner \frac{x}{p_1 p_2 \ldots p_r} \lrcorner.$$

by inclusion-exclusion. If we approximate this by ignoring the round downs the error is at most

$$1 + \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r,$$

and so

$$\pi(x) \leq r + x \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) + 2^r.$$

Now

$$\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} = \prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right).$$

If we expand the RHS we get the sum of the reciprocals of all numbers divisible by $p_1, p_2, \ldots, p_r$, which is at least

$$\sum_{k \leq n} \frac{1}{k} > \log n.$$

Thus

$$\pi(x) \leq r + \frac{x}{\log x} + 2^r.$$

If we take $r = \ulcorner \log x \urcorner$ then

$$\begin{aligned}
\pi(x) &\leq 2^{\log x + 2} + \frac{x}{\log\log x} && \text{take } r = \ulcorner \log x \urcorner \\
&= O\left(2^{\log x}\right) + \frac{x}{\log\log x} \\
&\leq o\left(\frac{x}{\log\log x}\right) + \frac{x}{\log\log x} && \text{as } \log 2 < 1 \\
&= O\left(\frac{x}{\log\log x}\right).
\end{aligned}$$