

9. PRIMES IN ARITHMETIC PROGRESSION

Definition 9.1. The **Riemann zeta-function** $\zeta(s)$ is the function which assigns to a real number $s > 1$ the convergent series

$$\sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Part of the significance of the Riemann zeta-function stems from

Theorem 9.2. If $s > 1$ then

$$\zeta(s) = \prod_p \left(\frac{1}{1 - p^{-s}} \right).$$

Proof. If we expand the RHS for all of the primes up to x we get

$$\prod_{p \leq x} \left(\frac{1}{1 - p^{-s}} \right) = \prod_{p \leq x} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right).$$

The product on the right is a finite product, over finitely many primes, of absolutely convergent geometric series. Thus we may rearrange the terms of the sum in any convenient order. If we expand the product we then get

$$\begin{aligned} \prod_{p \leq x} \left(\frac{1}{1 - p^{-s}} \right) &= \sum_{k: p|k \implies p \leq x} \frac{1}{k^s} \\ &= \sum_{k \leq x} \frac{1}{k^s} + \sum_{k > x: p|k \implies p \leq x} \frac{1}{k^s} \\ &= \Sigma_1(x) + \Sigma_2(x). \end{aligned}$$

As the series

$$\sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges, it follows that $\Sigma_1(x)$ converges to $\zeta(s)$ and $\Sigma_2(x)$ tends to zero. \square

Euler implicitly used the Riemann zeta-function to show that there are infinitely many primes by showing that the sum behind the LHS of (9.2) diverges at $s = 1$ (one can make this argument rigorous by taking the limit as s approaches one from above).

Dirichlet used the Riemann zeta-function to show that there are infinitely many primes in the arithmetic progression $an + b$ if and only if a and b are coprime. We will do the special case $a = 4$ and $b = 1$, we will show there are infinitely many primes of the form $4n + 1$.

In fact he introduced the notion of an L -function

$$L(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}$$

also known as a **Dirichlet series**.

We will need a simple variant on the Jacobi symbol

$$(-1/k) = \begin{cases} 0 & \text{if } 2|k \\ (-1)^{(k-1)/2} & k \text{ is odd.} \end{cases}$$

We can use this to define an L -function,

$$L(s) = \sum_{k=1}^{\infty} \frac{(-1/k)}{k^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \dots$$

Note that $(-1/p) = 1$ if and only if $p \equiv 1 \pmod{4}$, and so it is not so surprising that the Jacobi symbol turns up. Note that $(-1/k)$ is totally multiplicative, so that

$$(-1/kl) = (-1/k)(-1/l),$$

for all natural numbers k and l . It follows that if $k = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is the prime factorisation of n then

$$(-1/n) = (-1/p_1)^{e_1} (-1/p_2)^{e_2} \dots (-1/p_r)^{e_r}.$$

We have a simple generalisation of (9.2):

Lemma 9.3. *If a_k is a totally multiplicative sequence of numbers (meaning that the function $k \rightarrow a_k$ is totally multiplicative) and the series*

$$\sum a_k k^{-s}$$

converges absolutely for $s > s_0$ then

$$\sum \frac{a_k}{k^s} = \prod_p \left(1 - \frac{a_p}{p^s} \right)^{-1}$$

for $s > s_0$.

Proof. If we expand the RHS for all of the primes up to x we get

$$\prod_{p \leq x} \left(1 - \frac{a_p}{p^s} \right)^{-1} = \prod_{p \leq x} \left(1 + \frac{a_p}{p^s} + \frac{a_p^2}{p^{2s}} + \dots \right).$$

The product on the right is a finite product, over finitely many primes, of absolutely convergent geometric series. Thus we may rearrange the

terms of the sum in any convenient order. If we expand the product we then get

$$\begin{aligned} \prod_{p \leq x} \left(1 - \frac{a_p}{p^s}\right)^{-1} &= \sum_{k: p|k \implies p \leq x} \frac{a_k}{k^s} \\ &= \sum_{k \leq x} \frac{a_k}{k^s} + \sum_{k > x: p|k \implies p \leq x} \frac{a_k}{k^s} \\ &= \Sigma_1(x) + \Sigma_2(x). \end{aligned}$$

As the series

$$\sum_{k=1}^{\infty} \frac{a_k}{k^s}$$

converges, it follows that $\Sigma_1(x)$ converges to $\zeta(s)$ and $\Sigma_2(x)$ tends to zero. \square

Here is a simple case of Dirichlet's theorem:

Theorem 9.4. *There are infinitely many primes q of the form $4k + 1$; in fact there are so many that*

$$\lim_{s \rightarrow 1^+} \prod_q (1 - q^{-s})^{-1} = \infty.$$

Proof. We need to understand the behaviour of $\zeta(s)$ as s approaches 1 from above. We are going to apply the summation formula, (6.4), to

$$\lambda_n = n, \quad c_n = 1 \quad \text{and} \quad f(t) = \frac{1}{x^s}.$$

We have, for $s > 1$,

$$\begin{aligned} \sum_{k \leq x} \frac{1}{k^s} &= \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor t \rfloor}{t^{s+1}} dt \\ &= \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{t - \{t\}}{t^{s+1}} dt \\ &= \frac{\lfloor x \rfloor}{x^s} + \frac{s}{s-1} (1 - x^{1-s}) - s \int_1^x \frac{\{t\}}{t^{s+1}} dt. \end{aligned}$$

If we take the limit as x tends to infinity then we get

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt.$$

In fact the integral converges for $s > 0$. It follows that

$$\lim_{s \rightarrow 1^+} (s-1)\zeta(s) = 1 \quad \text{so that} \quad \lim_{s \rightarrow 1^+} \zeta(s) = \infty.$$

It is convenient to let q run over the primes congruent to 1 modulo 4 and for r to run over the primes congruent to -1 modulo 4. Since $L(s)$ is represented by a series that is absolutely convergent for $s > 1$ we have

$$\begin{aligned} L(s) &= \prod_p \left(1 - \frac{(-1/p)}{p^s}\right)^{-1} \\ &= \prod_q \left(1 - \frac{1}{q^s}\right)^{-1} \cdot \prod_r \left(1 + \frac{1}{r^s}\right)^{-1}. \end{aligned}$$

Now consider the product $\zeta(s)L(s)$. We have already shown that we can factor this into a product

$$\zeta(s)L(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \prod_q \left(1 - \frac{1}{q^s}\right)^{-2} \cdot \prod_r \left(1 - \frac{1}{r^{2s}}\right)^{-1},$$

valid for $s > 1$. Consider what happens as s approaches 1 from above. First the RHS. The first term approaches

$$\left(1 - \frac{1}{2}\right)^{-1} = 2.$$

For the last term we have

$$\begin{aligned} 0 &< \prod_r \left(1 - \frac{1}{r^{2s}}\right)^{-1} \\ &< \prod_r \left(1 - \frac{1}{r^2}\right)^{-1} \\ &< \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} \\ &= \zeta(2). \end{aligned}$$

Thus the last term is bounded. We want the middle term to go to infinity, so that the RHS goes to infinity.

Now consider the LHS. We already showed that $\zeta(s)$ goes to infinity. So if we can show that

$$\lim_{s \rightarrow 1^+} L(s) \neq 0$$

then the LHS goes to infinity, which forces the RHS to go to infinity.

We will show that $L(s)$ is continuous at $s = 1$ so that it suffices to show that $L(1) \neq 0$. Recall that the series $\sum u_k(s)$ is continuous, if u_1, u_2, \dots are continuous on a closed interval I and they converge uniformly on the interval I to the sum.

In our case

$$u_k(s) = \frac{1}{k^s},$$

are continuous on the whole s -axis and so we just need to check uniformity on closed intervals $[s_1, s_2]$, where $s_1 > 0$.

We have to show that given $\epsilon > 0$ there is an m_0 such that

$$\left| \sum_{k=m}^n \frac{(-1/k)}{k^s} \right| < \epsilon$$

for all $m > m_0$ and every $s \in I$.

The idea is to apply Abel's partial summation formula. Note first that

$$\left| \sum_{k=m}^n (-1/k) \right| \leq 1,$$

regardless of m and n . If we put $A_{m-1} = 0$ and

$$A_k = \sum_{l=m}^k (-1/l)$$

for $k \geq m$ then

$$\begin{aligned} \left| \sum_{k=m}^n \frac{(-1/k)}{k^s} \right| &= \left| \sum_{k=m}^n \frac{A_k - A_{k-1}}{k^s} \right| \\ &= \left| \sum_{k=m}^n A_k (k^{-s} - (k+1)^{-s}) + A_n (n+1)^{-s} - A_{m-1} m^{-s} \right| \\ &\leq \sum_{k=m}^n (k^{-s} - (k+1)^{-s}) + (n+1)^{-s} \\ &= m^{-s} \\ &\leq m^{-s_1}. \end{aligned}$$

On the other hand, $m^{-s_1} < \epsilon$ for all m sufficiently large.

Finally, note that

$$\begin{aligned} L(1) &= 1 - 1/3 + 1/5 - 1/7 + 1/9 + \dots \\ &= (1 - 1/3) + (1/5 - 1/7) + (1/9 - 1/11) + \dots \\ &> 2/3. \end{aligned}$$

□