8. Better bounds

Theorem 8.1. *There are positive constants c_3 and c_4 such that*

$$c_3 r \log r < p_r < c_4 r \log r,$$

for $r > 1$.

Proof. If we apply (7.1) with $x = p_r$ then we get

$$c_1 \frac{p_r}{\log p_r} < r < c_2 \frac{p_r}{\log p_r}.$$

The second inequality gives

$$\begin{aligned} p_r &> c_3 r \log p_r \\ &> c_3 r \log r, \end{aligned}$$

for all r. For the first inequality we get

$$\frac{\log p_r}{\sqrt{p_r}} < c_1 < \frac{r \log p_r}{p_r}$$

since the LHS tends to zero. Since the first term is less than the third, it follows that $p_r < r^2$, so that $\log p_r < 2r$. Therefore

$$p_r < \frac{1}{c_1} r \cdot 2 \log r,$$

for r sufficiently large. Thus $p_r < c_4 r \log r$ for all $r > 1$. □

Theorem 8.2. *We have*

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

Furthermore, there is a constant C such that

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + C + O\left(\frac{1}{\log x}\right).$$

Proof. In the course of the proof of (6.3) we showed

$$n \sum_{p \leq x} \frac{\log p}{p} = n \log n + O(n) + O(\pi(n) \log n).$$

Dividing by n gives the first result.

For the second result, following the course of the proof of (6.5), and using the better bound for the first sum, we get

$$\sum_{p\le x}\frac{1}{p} \le \frac{1}{\log x}(\log x + O(1)) + \int_2^x \log t \frac{\mathrm{d}t}{t\log^2 t} + \int_2^x \left(\sum_{p\le t}\frac{\log p}{p} - \log t\right)\frac{\mathrm{d}t}{t\log^2 t}$$

$$= 1 + O\left(\frac{1}{\log x}\right) + \log\log x - \log\log 2$$

$$+ \int_2^\infty \left(\sum_{p\le t}\frac{\log p}{p} - \log t\right)\frac{\mathrm{d}t}{t\log^2 t} + \int_x^\infty \frac{O(1)\mathrm{d}t}{t\log^2 t}.$$

The first integral converges, as

$$\int_2^\infty \frac{\mathrm{d}t}{t\log^2 t}$$

is a convergent integral and the second integral is

$$O\left(\frac{1}{\log x}\right). \qquad \square$$