7. True order of $\pi(x)$

Theorem 7.1. *There are positive constants c_1 and c_2 such that for $x \geq 2$*

$$c_1 \frac{x}{\log x} < \pi(x) < c_2 \frac{x}{\log x}.$$

The prime number theorem states that in fact

$$\pi(x) \sim \frac{x}{\log x}.$$

Let

$$\rho(x) = \frac{\pi(x)}{\frac{x}{\log x}},$$

the ratio between the two functions. Using a homework problem, one can show

$$c_1 < \liminf \rho(x) \leq 1 \leq \limsup \rho(x) < c_2.$$

The prime number theorem states that both the liminf and the limsup are equal to one.

(7.1) was proved by Chebyshev, who proved that $c_1 > 0.92\ldots$ and $c_2 < 1.105\ldots$. One can compute the constants in the proof of (7.1) and get slightly worse bounds than Chebyshev's.

Chebyshev is famous for an inequality named after him. Analysis is all about comparing different quantities. Most of the time one just applies the triangle inequality but Chebyshev's inequality is a fundamentally different type of inequality.

The idea behind the proof of (7.1) is to find an expression involving factorials such that the exponent of each prime up to n, in the prime factorisation of the expression, is close to one. We are going to take a slightly different expression than the one Chebyshev used, namely

$$\binom{2n}{n} = \frac{2n!}{n!n!}.$$

We introduce the notation

$$f(x) \ll g(x) \qquad \text{as a substitute for} \qquad f(x) = O(g(x)).$$

Proof of (7.1). Pick $n \geq 2$. For each prime $p \leq 2n$ there is a unique integer r_p such that

$$p^{r_p} \leq 2n < p^{r_p+1}.$$

Suppose that $n < p \leq 2n$. Then $p|(2n)!$ but p does not divide $n!$. Thus

$$\left(\prod_{n<p\leq 2n} p \right) \Bigg| \binom{2n}{n}.$$

Now suppose that $p \leq 2n$. Then the highest power of p which divides $(2n)!$ has exponent

$$\sum_{m=1}^{r_p} \lfloor \frac{2n}{p^m} \rfloor.$$

On the other hand the highest power of p which divides $(n!)^2$ has exponent

$$2\sum_{m=1}^{r_p} \lfloor \frac{n}{p^m} \rfloor.$$

Thus the highest power of p which divides $\binom{2n}{n}$ has exponent

$$\begin{aligned} \sum_{m=1}^{r_p} \left(\lfloor \frac{2n}{p^m} \rfloor - 2\lfloor \frac{n}{p^m} \rfloor \right) &\leq \sum_{m=1}^{r_p} 1 \\ &= r_p. \end{aligned}$$

It follows that

$$\binom{2n}{n} \Bigg| \left(\prod_{p \leq 2n} p^{r_p} \right).$$

Putting all of this together we have

$$n^{\pi(2n)-\pi(n)} \leq \prod_{n<p\leq 2n} p \leq \binom{2n}{n} \leq \prod_{p\leq 2n} p^{r_p} \leq (2n)^{\pi(2n)}.$$

Taking logs gives

$$(\pi(2n) - \pi(n)) \log n \leq \log \binom{2n}{n} \leq \pi(2n) \log(2n).$$

Now

$$\begin{aligned} \binom{2n}{n} &\leq \sum_{k=0}^{2n} \binom{2n}{k} \\ &= 2^{2n}. \end{aligned}$$

On the other hand

$$\begin{aligned}\binom{2n}{n} &= \frac{(n+1)\cdot(n+2)\dots 2n}{1\cdot 2\cdot 3\dots n}\\ &= \prod_{a=1}^{n}\frac{n+a}{a}\\ &\geq \prod_{a=1}^{n} 2\\ &= 2^n.\end{aligned}$$

It follows that

$$(\pi(2n)-\pi(n))\log n \leq 2n\log 2$$

and

$$\pi(2n)\log 2n \geq n\log 2.$$

Thus

$$\pi(2n) \gg \frac{n}{\log n}.$$

Thus

$$\begin{aligned}\pi(x) &\geq \pi(2\llcorner\frac{x}{2}\lrcorner)\\ &\gg \frac{\llcorner x/2\lrcorner}{\log\llcorner x/2\lrcorner}\\ &\gg \frac{x}{\log x}.\end{aligned}$$

As $\pi(x) \geq 1$ for $x \geq 2$, it follows that

$$\pi(x) > c_1\frac{x}{\log x}$$

for $x \geq 2$.

On the other hand, if $y \geq 4$, we have

$$\begin{aligned}\pi(y)-\pi(y/2) &= \pi(y)-\pi(\llcorner\frac{y}{2}\lrcorner)\\ &\leq 1+\pi(2\llcorner\frac{y}{2}\lrcorner)-\pi(\llcorner\frac{y}{2}\lrcorner)\\ &\ll \frac{\llcorner\frac{y}{2}\lrcorner}{\log\llcorner\frac{y}{2}\lrcorner}\\ &\ll \frac{y}{\log y}.\end{aligned}$$

It follows that

$$\pi(y)-\pi(y/2) \leq c\frac{y}{\log y},$$

for $y \geq 2$, since it is easy to see the inequality holds over the range $2 \leq y \leq 4$. As

$$\pi(y/2) \leq y/2$$

we get

$$\begin{aligned}\pi(y)\log y - \pi\left(\frac{y}{2}\right)\log\frac{y}{2} &= \left(\pi(y) - \pi\left(\frac{y}{2}\right)\right)\log y + \pi\left(\frac{y}{2}\right)\log 2\\ &< c\frac{y}{\log y}\cdot\log y + \frac{y}{2}\\ &< c'y.\end{aligned}$$

If $m \geq 0$ is an integer such that

$$y = \frac{x}{2^m} \qquad \text{with} \qquad 2^m \leq \frac{x}{2}$$

then we get

$$\pi\left(\frac{x}{2^m}\right)\log\frac{x}{2^m} - \pi\left(\frac{x}{2^{m+1}}\right)\log\frac{x}{2^{m+1}} < c'\frac{x}{2^m}.$$

Summing over all such m we get

$$\pi(x)\log x - \pi\left(\frac{x}{2^{\mu+1}}\right)\log\frac{x}{2^{\mu+1}} < 2c'x,$$

where μ is defined by the inequalities

$$2^\mu \leq \frac{x}{2} < 2^{\mu+1}.$$

It follows that

$$\frac{x}{2^{\mu+1}} < 2,$$

so that

$$\pi\left(\frac{x}{2^{\mu+1}}\right) = 0.$$

Thus

$$\pi(x)\log x < c_2 x. \qquad \square$$

We end with some parenthetical remarks about the liminf and limsup. It is probably easiest to work only with sequences.

Let $a_1, a_2, \dots$ be a squence of real numbers. We define two auxiliary seqeunces $b_1, b_2, \dots$ and $c_1, c_2, \dots$ associated to $a_1, a_2, \dots$.

$$b_i = \inf_{j\geq i} a_j \qquad \text{and} \qquad c_i = \sup_{j\geq i} a_j.$$

Example 7.2. *Suppose that $a_1, a_2, \dots$ is the sequence*

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even.}\end{cases}$$

Then $b_n = -1$ and $c_n = 1$ are constant sequences.

Example 7.3. *Suppose that $a_1, a_2, \dots$ is the sequence*

$$a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ -\frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

Then

$$b_1, b_2, \dots = -1/2, -1/2, -1/4, -1/4, -1/6, -1/6, \dots \qquad \text{and}$$
$$c_1, c_2, \dots = 1, 1/3, 1/3, 1/5, 1/5, 1/7, \dots.$$

Example 7.4. *Let $a_1, a_2, \dots$ be the sequence $a_n = n$.*

Then $b_n = n$ and $c_n = \infty$.

Definition-Proposition 7.5. *The limit (possibly infinite) b and c of the sequences $b_1, b_2, \dots$ and $c_1, c_2, \dots$ exist; $b = \liminf a_n$ is called the **liminf** and $c = \liminf a_n$ is called the **limsup**.*

$a_1, a_2, \dots$ converges if and only if $b = c$ in which case $a = \lim a_n = b$.

Proof. Note that

$$\begin{aligned} b_n &= \inf_{m \geq n} a_m \\ &= \min(a_n, \inf_{m \geq n+1} a_m) \\ &= \min(a_n, b_{n+1}) \\ &\leq b_{n+1}. \end{aligned}$$

Thus $b_1, b_2, \dots$ is monotonic increasing. By symmetry, $c_1, c_2, \dots$ is monotonic decreasing. Thus the limits exist.

Note that from $b_n \leq \min(a_n, b_{n+1})$ one gets by symmetry

$$b_n \leq a_n \leq c_n.$$

Thus if $b = c$ then $a_1, a_2, \dots$ converges to b.

Now suppose that $a_1, a_2, \dots$ converges to a. Fix $\epsilon > 0$. Then we may find $n \geq n_0$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$. In particular $a_n > a - \epsilon$. Therefore $b_n \geq a - \epsilon$ and so $b_1, b_2, \dots$ converges to a. By symmetry, $c_1, c_2, \dots$ converges to a. □