## 3. Round down and fractional part

**Definition 3.1.** *Let $x \in \mathbb{R}$ be a real number.*

*The **round down** of $x$, denoted $\llcorner x \lrcorner$, is the largest integer smaller than $x$. The **fractional part** of $x$, denoted $\{x\}$, is the difference $x - \llcorner x \lrcorner$.*

For example,

$$\begin{aligned} \llcorner \sqrt{2} \lrcorner &= 1 \qquad \text{and} \qquad \{\sqrt{2}\} = 0.4142135\dots \\ \llcorner e \lrcorner &= 2 \qquad \text{and} \qquad \{e\} = 0.71828\dots \\ \llcorner \pi \lrcorner &= 3 \qquad \text{and} \qquad \{\pi\} = 0.14159\dots \\ \llcorner 9/2 \lrcorner &= 4 \qquad \text{and} \qquad \{9/2\} = 1/2. \end{aligned}$$

The round down satisfies some easy basic properties:

**Proposition 3.2.** *Let $x$, $x_1$ and $x_2$ be real numbers and let $a$ and $n$ be integers. Assume that $n$ is a natural number in (5), (8) and (9).*

(1) $x = \llcorner x \lrcorner + \{x\}$.

(2) $\llcorner x + n \lrcorner = \llcorner x \lrcorner + n$.

(3)
$$\llcorner x \lrcorner + \llcorner -x \lrcorner = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise.} \end{cases}$$

(4)
$$\llcorner x_1 \lrcorner + \llcorner x_2 \lrcorner \leq \llcorner x_1 + x_2 \lrcorner.$$

(5)
$$\llcorner \frac{x}{n} \lrcorner = \llcorner \frac{\llcorner x \lrcorner}{n} \lrcorner.$$

(6)
$$0 \leq \llcorner x \lrcorner - 2\llcorner \frac{x}{2} \lrcorner \leq 1.$$

(7) *The cardinality of*
$$\{ m \in \mathbb{Z} \mid x_1 < m \leq x_2 \}$$
*is* $\llcorner x_2 \lrcorner - \llcorner x_1 \lrcorner$.

(8) *The cardinality of*
$$\{ k \in \mathbb{N} \mid kn \leq x \}$$
*is* $\llcorner \frac{x}{n} \lrcorner$.

(9)
$$n\{ \frac{a}{n} \}$$
*is the residue of $a$ modulo $n$ between $0$ and $n-1$.*

*Proof.* (1) is immediate from the definitions. By definition of the round down

$$\llcorner x \lrcorner \leq x$$

so that adding $n$ to both sides we get

$$\llcorner x \lrcorner + n \leq x + n.$$

Therefore $\llcorner x \lrcorner + n$ is an integer less than $x + n$. As $\llcorner x + n \lrcorner$ is the largest integer less than $x + n$, we have

$$\llcorner x \lrcorner + n \leq \llcorner x + n \lrcorner.$$

On the other hand,

$$\llcorner x + n \lrcorner \leq x + n$$

so that subtracting $n$ from both sides, we get

$$\llcorner x + n \lrcorner - n \leq x.$$

The LHS is an integer less than $x$ so that

$$\llcorner x + n \lrcorner - n \leq \llcorner x \lrcorner.$$

Adding $n$ to both sides, we get

$$\llcorner x + n \lrcorner \leq \llcorner x \lrcorner + n.$$

Since we have an equality both ways, this gives (2).

If $x$ is an integer then so is $-x$ and so $\llcorner x \lrcorner = x$ and $\llcorner -x \lrcorner = -x$. But then

$$\llcorner x \lrcorner + \llcorner -x \lrcorner = x - x = 0.$$

Otherwise neither $x$ nor $-x$ is an integer, so that $\llcorner x \lrcorner < x$ and $\llcorner -x \lrcorner < -x$. In particular

$$\llcorner x \lrcorner + \llcorner -x \lrcorner < x - x = 0.$$

But $x - \llcorner x \lrcorner < 1$ and $-x - \llcorner -x \lrcorner < 1$ so that

$$\begin{aligned} \llcorner x \lrcorner + \llcorner -x \lrcorner &> (x - 1) + (-x - 1) \\ &= -2. \end{aligned}$$

Since $\llcorner x \lrcorner + \llcorner -x \lrcorner$ is an integer between $-2$ and $0$ it must be $-1$. This is (3).

We have

$$\llcorner x_1 \lrcorner \leq x_1 \qquad \text{and} \qquad \llcorner x_2 \lrcorner \leq x_2,$$

so that

$$\llcorner x_1 \lrcorner + \llcorner x_2 \lrcorner \leq x_1 + x_2.$$

As the LHS is an integer, this gives (4).

As

$$\llcorner x \lrcorner \leq x$$

we have

$$\frac{\lfloor x \rfloor}{n} \le \frac{x}{n},$$

and so

$$\lfloor \frac{\lfloor x \rfloor}{n} \rfloor \le \lfloor \frac{x}{n} \rfloor.$$

On the other hand,

$$\lfloor \frac{x}{n} \rfloor \le \frac{x}{n}.$$

Thus

$$n \lfloor \frac{x}{n} \rfloor \le x.$$

As the LHS is an integer, it follows that

$$n \lfloor \frac{x}{n} \rfloor \le \lfloor x \rfloor,$$

so that

$$\lfloor \frac{x}{n} \rfloor \le \frac{\lfloor x \rfloor}{n}.$$

As the LHS is an integer, it follows that

$$\lfloor \frac{x}{n} \rfloor \le \lfloor \frac{\lfloor x \rfloor}{n} \rfloor.$$

As we have equality both ways this gives (5).

If we apply (5) with $n = 2$ we see that

$$\begin{aligned} 2 \lfloor \frac{x}{2} \rfloor &= 2 \lfloor \frac{\lfloor x \rfloor}{2} \rfloor \\ &= 2 \frac{\lfloor x \rfloor}{2} - 2 \{ \frac{\lfloor x \rfloor}{2} \} \\ &= \lfloor x \rfloor - 2 \{ \frac{\lfloor x \rfloor}{2} \}. \end{aligned}$$

Thus

$$\lfloor x \rfloor - 2 \lfloor \frac{x}{2} \rfloor = 2 \{ \frac{\lfloor x \rfloor}{2} \}.$$

The LHS is an integer and the RHS belongs to the interval $[0, 2)$. The LHS is either 0 or 1. This is (6).

Note that we have an equality of the two sets

$$\{ m \in \mathbb{Z} \,|\, x_1 < m \le x_2 \} = \{ m \in \mathbb{Z} \,|\, \lfloor x_1 \rfloor + 1 \le m \le \lfloor x_2 \rfloor \}.$$

This gives (7).

Note that we have an equality of the two sets

$$\{ k \in \mathbb{N} \,|\, kn \le x \} = \{ k \in \mathbb{N} \,|\, k \le \frac{x}{n} \}.$$

This gives (8).

We may write

$$a = qn + r,$$

where $q$ and $r$ are integers and $0 \leq r \leq n-1$. Note that $r$ is equivalent to $a$ modulo $n$ and $r$ is between 0 and $n-1$. If we divide through by $n$ we get

$$\frac{a}{n} = q + \frac{r}{n}.$$

Note that $q$ is an integer less than $\frac{a}{n}$. Thus

$$q \leq \lfloor \frac{a}{n} \rfloor.$$

On the other hand

$$\frac{a}{n} = \lfloor \frac{a}{n} \rfloor + \{ \frac{a}{n} \}.$$

Multiplying through by $n$ we get

$$a = \lfloor \frac{a}{n} \rfloor n + \{ \frac{a}{n} \} n.$$

As the first two terms are integers it follows that the last term is a non-negative integer. Thus

$$q \geq \lfloor \frac{a}{n} \rfloor.$$

so that

$$q = \lfloor \frac{a}{n} \rfloor \qquad \text{and} \qquad r = \{ \frac{a}{n} \} n.$$

This gives (9). □

**Proposition 3.3.** *Let $p$ be a prime.*

*If $n$ is a natural number and $n! = p^r m$ where $m$ is coprime to $p$ then*

$$r = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots.$$

*Proof.* Note that

$$r = \sum_{i=1}^{\infty} i r_i,$$

where $r_i$ is the number of integers from 1 to $n$ divisible by $p^i$ but not $p^{i+1}$. Let $s_i$ be the number of integers from 1 to $n$ divisible by $p^i$. Note that $r_i = s_i - s_{i+1}$ so that

$$\begin{aligned}
r &= \sum_{i=1}^{\infty} i r_i \\
&= \sum_{i=1}^{\infty} i(s_i - s_{i+1}) \\
&= \sum_{i=1}^{\infty} i s_i - \sum_{i=1}^{\infty} i s_{i+1} \\
&= \sum_{i=1}^{\infty} i s_i - \sum_{i=1}^{\infty} (i-1) s_i \\
&= \sum_{i=1}^{\infty} s_i.
\end{aligned}$$

On the other hand, of the numbers from 1 to $n$, there are

$$s_i = \lfloor \frac{n}{p^i} \rfloor$$

numbers divisible by $p^i$. $\square$