17. This and that

The material in this section is covered in the book "Introduction to analytic number theory" by Apostol.

Definition 17.1. *If p and q are two lattice points then we say that p and q are (mutually)* ***visible*** *if the only lattice points on the line segment connecting p and q are p and q themselves.*

Lemma 17.2. *Two lattice points (a, b) and (c, d) are visible if and only if $a - c$ and $b - d$ are coprime.*

Proof. There is no loss in generality in assuming that $(c, d) = (0, 0)$ is the origin.

Let d be the greatest common divisor of a and b. We have to show that (a, b) is invisible from the origin if and only if $d > 1$.

Suppose that $d > 1$. Then we may find l and m such that $a = dl$ and $b = dm$. Then (l, m) is on the line through the origin and (a, b) and comes before (a, b), so that (a, b) is not visible from the origin.

Now suppose that (l, m) a lattice point on the line between the origin and (a, b). Then there is a real number $t \in (0, 1)$ such that $(l, m) = t(a, b)$. As l and m are integers, it follows that t is a rational number. Suppose that $t = \lambda/\mu$ with $(\lambda, \mu) = 1$. As $t < 1$ it follows that $\mu > 1$.

It follows that

$$\lambda a = \mu l \qquad \text{and} \qquad \lambda b = \mu m.$$

Thus μ divides both a and b, so that $d > 1$. □

Note that there are infinitely many points which are visible from the origin. It is natural to wonder about their distribution. Consider the square region centred around the origin,

$$|x| \leq r \qquad \text{and} \qquad |y| \leq r.$$

Let $N(r)$ be the number of lattice points in this square and let $N'(r)$ be the number of visible lattice points in this square. The quotient

$$\frac{N'(r)}{N(r)}$$

is the ratio of visible lattice points to total number of lattice points inside the square. The limit as r goes to infinity is called the **density** of lattice points visible from the origin.

Theorem 17.3. *The set of lattice points visible from the origin has density*

$$\frac{6}{\pi^2}.$$

Proof. We have to prove that

$$\lim_{r\to\infty} \frac{N'(r)}{N(r)} = \frac{6}{\pi^2}.$$

First note that the eight closest points to the origin are all visible:

$$(\pm 1, 0) \qquad (0, \pm 1) \qquad \text{and} \qquad (\pm 1, \pm 1).$$

Otherwise we just have to count the number of visible points in the region

$$\{\, (x, y) \in \mathbb{N}^2 \mid 2 \le x \le r, 1 \le y \le x \,\}$$

and multiply by 8. Thus

$$\begin{aligned} N'(r) &= 8 + 8 \sum_{2\le n\le r} \sum_{\substack{1\le m<n \\ (m,n)=1}} 1 \\ &= 8 + 8 \sum_{2\le n\le r} \varphi(n). \end{aligned}$$

Using our knowledge of the average value of φ, this gives

$$N'(r) = \frac{4r^2}{\zeta(2)} + O(r \log r).$$

On the other hand, the total number of lattice points in the square is

$$\begin{aligned} N(r) &= 4 \sum_{n\le r} \sum_{m\le r} 1 \\ &= 4 \llcorner r \lrcorner \llcorner r \lrcorner \\ &= 4r^2 + O(r). \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to\infty} \frac{N'(r)}{N(r)} &= \frac{\frac{4r^2}{\zeta(2)} + O(r\log r)}{4r^2 + O(r)} \\ &= \frac{\frac{4}{\zeta(2)} + O(\log r/r)}{4 + O(1/r)} \\ &= \frac{1}{\zeta(2)} \\ &= \frac{6}{\pi^2}. \end{aligned}$$

□

One way to state (17.3) is to say that a lattice point chosen at random has probability $6/\pi^2$ of being visible. In light of (17.2), this is the same as the probability two integers chosen at random are coprime.

We have already seen that dealing with the difference between x and its round down is one of the key technical parts of analytic number theory. There are a couple of general results that state that sometimes it is automatic we can interchange the two.

Theorem 17.4. *Let*

$$a_1, a_2, \dots$$

be a sequence of non-negative reals. Suppose that

$$\sum_{n\leq x} a_n \llcorner \frac{x}{n} \lrcorner = x\log x + O(x) \qquad \text{for all} \qquad x \geq 1.$$

Then

(1) *For $x \geq 1$ we have*

$$\sum_{n\leq x} \frac{a_n}{n} = \log x + O(1).$$

(2) *There is a constant $B > 0$ such that*

$$\sum_{n\leq x} a_n \leq Bx$$

for all $x \geq 1$.

(3) *There is a constant $A > 0$ and an x_0 such that*

$$\sum_{n\leq x} a_n \geq Ax$$

for all $x \geq x_0$.

Proof. We define two auxiliary functions,

$$P(x) = \sum_{n\leq x} a_n \qquad \text{and} \qquad Q(x) = \sum_{n\leq x} a_n \llcorner \frac{x}{n} \lrcorner.$$

We first prove (2).

Claim 17.5.

$$P(x) - P\left(\frac{x}{2}\right) \leq Q(x) - 2Q\left(\frac{x}{2}\right).$$

Proof of (17.5). We have

$$\begin{aligned} Q(x) - 2Q\left(\frac{x}{2}\right) &= \sum_{n\leq x} \llcorner \frac{x}{n} \lrcorner a_n - 2\sum_{n\leq x/2} \llcorner \frac{x}{2n} \lrcorner a_n \\ &= \sum_{n\leq x/2} \left(\llcorner \frac{x}{n} \lrcorner - 2\llcorner \frac{x}{2n} \lrcorner \right) a_n + \sum_{x/2<n\leq x} \llcorner \frac{x}{n} \lrcorner a_n. \end{aligned}$$

As $\lfloor 2y \rfloor - 2\lfloor y \rfloor$ is either 0 or 1, the first sum is non-negative, so that

$$\begin{aligned} Q(x) - 2Q\left(\frac{x}{2}\right) &\geq \sum_{x/2<n\leq x} \left\lfloor \frac{x}{n} \right\rfloor a_n \\ &= \sum_{x/2<n\leq x} a_n \\ &= P(x) - P\left(\frac{x}{2}\right). \end{aligned} \qquad \square$$

By assumption

$$\begin{aligned} Q(x) - 2Q\left(\frac{x}{2}\right) &= x\log x + O(x) - 2\left(\frac{x}{2}\log\frac{x}{2} + O(x)\right) \\ &= O(x). \end{aligned}$$

Thus (17.5) implies that

$$P(x) - P\left(\frac{x}{2}\right) = O(x).$$

Hence there is a constant $K > 0$ such that

$$P(x) - P\left(\frac{x}{2}\right) \leq Kx,$$

for all $x \geq 1$. Replacing x by $x/2$ and by $x/4$, etc, we get

$$\begin{aligned} P\left(\frac{x}{2}\right) - P\left(\frac{x}{4}\right) &\leq K\frac{x}{2}, \\ P\left(\frac{x}{4}\right) - P\left(\frac{x}{8}\right) &\leq K\frac{x}{4}, \end{aligned}$$

and so on. Note that

$$P\left(\frac{x}{2^n}\right) = 0$$

when $2^n > x$. Adding all of these inequalities together, we get

$$\begin{aligned} P(x) &\leq Kx\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) \\ &= 2Kx. \end{aligned}$$

This establishes (2), with $B = 2K$.

We now turn to (1). We have

$$\begin{aligned} Q(x) &= \sum_{n\le x} \lfloor \frac{x}{n} \rfloor a_n \\ &= \sum_{n\le x} \left(\frac{x}{n} + O(1)\right) a_n \\ &= x\sum_{n\le x} \frac{a_n}{n} + O\left(\sum_{n\le x} a_n\right) \\ &= x\sum_{n\le x} \frac{a_n}{n} + O(x). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n\le x} \frac{a_n}{n} &= \frac{1}{x}Q(x) + O(1) \\ &= \log x + O(1). \end{aligned}$$

This is (1).

We now prove (3). Let

$$A(x) = \sum_{n\le x} \frac{a_n}{n}.$$

Then (1) says that

$$A(x) = \log x + R(x),$$

where $R(x)$ is the error term. As $R(x) = O(x)$ we have $R(x) \le Mx$ for some $M > 0$.

Pick $0 < \alpha < 1$ (we will be more precise about α later) and consider

$$\begin{aligned} A(x) - A(\alpha x) &= \sum_{n\le x} \frac{a_n}{n} - \sum_{n\le \alpha x} \frac{a_n}{n} \\ &= \sum_{\alpha x < n\le x} \frac{a_n}{n}. \end{aligned}$$

If $x \ge 1$ and $\alpha x \ge 1$ then we have

$$\begin{aligned} A(x) - A(\alpha x) &= \log x + R(x) - (\log \alpha x + R(\alpha x)) \\ &= -\log\alpha + R(x) - R(\alpha x) \\ &\ge -\log\alpha - |R(x)| - |R(\alpha x)| \\ &\ge -\log\alpha - 2M. \end{aligned}$$

Now pick α such that $-\log\alpha - 2M = 1$. This implies that

$$\log\alpha = -2M - 1 \qquad \text{so that} \qquad \alpha = e^{-2M-1}.$$

Note that $0 < \alpha < 1$, as required. With this choice of α we have the inequality

$$A(x) - A(\alpha x) \geq 1 \qquad \text{if} \qquad x \geq \frac{1}{\alpha}.$$

On the other hand

$$\begin{aligned} A(x) - A(\alpha x) &= \sum_{\alpha x < n \leq x} \frac{a_n}{n} \\ &\leq \frac{1}{\alpha x} \sum_{n \leq x} a_n \\ &= \frac{P(x)}{\alpha x}. \end{aligned}$$

Therefore

$$\frac{P(x)}{\alpha x} \geq 1 \qquad \text{if} \qquad x \geq \frac{1}{\alpha}.$$

Thus

$$P(x) \geq \alpha x \qquad \text{if} \qquad x \geq \frac{1}{\alpha},$$

which is (3) with $A = \alpha$ and $x_0 = \frac{1}{\alpha}$. □