16. Bernoulli numbers

The material in this section is covered in the books "An introduction to the theory of numbers" by Hardy and Wright and "A course in number theory" by Rose.

Definition 16.1. *The sequence of rational numbers*

$$B_0, B_1, \dots,$$

called the ***Bernoulli numbers*** *are defined using the power series expansion*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

which is valid for $|z| < 2\pi$.

Lemma 16.2. *The Bernoulli numbers satisfy the recursion* $B_0 = 1$ *and*

$$(m+1)B_m = -\sum_{j=0}^{m-1} \binom{m+1}{j} B_j.$$

Proof. If we apply L'Hôpital then we see that

$$\begin{aligned} B_0 &= \lim_{z \to 0} \frac{z}{e^z - 1} \\ &= \lim_{z \to 0} \frac{1}{e^z} \\ &= 1. \end{aligned}$$

On the other hand, if we multiply

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

by $e^z - 1$, then we get

$$\begin{aligned} z &= \sum_{m=1}^{\infty} \frac{z^m}{m!} \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \\ &= \sum_{m=1, n=0}^{\infty} B_n \frac{z^{m+n}}{m!n!} \\ &= \sum_{k=0}^{\infty} \sum_{\substack{m>0 \\ k+1=m+n}} B_n \frac{z^{k+1}}{m!n!}. \end{aligned}$$

Once again this gives $B_0 = 1$ and equating coefficients of $k+1 = m+n$ we see that

$$\frac{1}{(k+1)!}\sum_{j=0}^{k}\binom{k+1}{j}B_j = 0. \qquad \square$$

Using the recursive definition we can compute the first few terms

$$\begin{aligned} 2B_1 &= -1 \\ 3B_2 &= -3B_1 - 1 \\ 4B_3 &= -6B_2 - 4B_1 - 1, \end{aligned}$$

so that

$$B_1 = -\frac{1}{2} \qquad B_2 = \frac{1}{6} \qquad B_3 = 0 \qquad B_4 = -\frac{1}{30} \qquad B_5 = 0 \qquad B_6 = \frac{1}{42} \dots.$$

Note after B_1 the odd terms are zero and the even terms alternate.

Lemma 16.3. *$B_{2n+1} = 0$ for $n > 0$.*

Proof. As

$$B_2 = -\frac{1}{2}.$$

we have

$$\begin{aligned} 1 + \sum_{n=2}^{\infty} B_n \frac{z^n}{n!} &= \frac{z}{e^z - 1} + \frac{z}{2} \\ &= \frac{z(e^z + 1)}{2(e^z - 1)}. \end{aligned}$$

Now use the fact that the last function is invariant under replacing z by $-z$, so that it is even. $\square$

Bernoulli introduced his numbers because of

Proposition 16.4. *If k is a natural number then*

$$1^k + 2^k + \dots + (n-1)^k = \sum_{r=0}^{k} \frac{1}{k+1-r}\binom{k}{r} n^{k+1-r} B_r.$$

Proof. The LHS is the coefficient of z^{k+1} in

$$\begin{aligned} k!z(1 + e^z + e^{2z} + \dots + e^{(n-1)z}) &= k!z\frac{1 - e^{nz}}{1 - e^z} \\ &= k!\frac{z}{e^z - 1}(e^{nz} - 1) \\ &= k!\left(B_0 + \frac{B_1}{1!}z + \dots + \frac{B_n}{n!}z^n + \dots\right)\left(nz + \frac{n^2z^2}{2} + \dots\right) \end{aligned}$$

and the result follows if we expand the RHS and extract the coefficient of z^{k+1}. $\square$

Theorem 16.5 (Euler)**.** *If n is a natural number then*

$$2(2n)!\zeta(2n) = (-1)^{n+1}(2\pi)^{2n}B_{2n}.$$

Proof. Following Euler we will assume that the sine function behaves just like a polynomial, which is to say that the sine function is the product of the linear factors, one for each root up to a constant factor:

$$\begin{aligned}\sin z &= z\prod_{n=1}^{\infty}\left(1-\frac{z}{n\pi}\right)\prod_{n=1}^{\infty}\left(1+\frac{z}{n\pi}\right)\\ &= z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right).\end{aligned}$$

(in fact this is valid but needs some additional justification). If we take the logarithmic derivative of both sides then we get

$$\cot z = \frac{1}{z} - 2\sum_{n=1}^{\infty}\frac{z}{n^2\pi^2 - z^2}.$$

Realising each term in the product as a geometric series and expanding we get

$$\begin{aligned}z\cot z &= 1 - 2\sum_{n=1}^{\infty}\frac{z^2}{n^2\pi^2 - z^2}\\ &= 1 - 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{z}{n\pi}\right)^{2m}\\ &= 1 - 2\sum_{m=1}^{\infty}\left(\frac{z}{\pi}\right)^{2m}\sum_{n=1}^{\infty}\frac{1}{n^{2m}}\\ &= 1 - 2\sum_{m=1}^{\infty}\zeta(2m)\left(\frac{z}{\pi}\right)^{2m}.\end{aligned}$$

As

$$\begin{aligned}e^{iz} &= \cos z + i\sin z\\ e^{-iz} &= \cos z - i\sin z,\end{aligned}$$

it follows that

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \qquad \text{and} \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

Therefore

$$\cot z = i\frac{(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})}.$$

It follows that

$$\begin{aligned}
z \cot z &= iz\frac{(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})} \\
&= iz\frac{(e^{2iz} + 1)}{(e^{2iz} - 1)} \\
&= iz + \frac{2iz}{(e^{2iz} - 1)} \\
&= 1 + \sum_{m=2}^{\infty} B_m \frac{(2iz)^m}{m!}.
\end{aligned}$$

Equating coefficients of z^{2m}, $m > 0$, we get

$$-2\frac{\zeta(2m)}{\pi^{2m}} = (-1)^m 2^{2m} \frac{B_{2m}}{(2m)!}.$$ □

Corollary 16.6.

(1) If $m > 0$ then

$$(-1)^{m+1} B_{2m} > 0.$$

(2)

$$|B_{2m}| > 2\left(\frac{m}{\pi e}\right)^{2m}.$$

Proof. (1) is immediate from (16.5) and the fact that $\zeta(2m) > 0$.

(2) follows as

$$e^n > \frac{n^n}{n!}.$$ □

Corollary 16.7.

$$\zeta(2) = \frac{\pi^2}{6}.$$

Proof.

$$\begin{aligned}
\zeta(2) &= 2\pi^2 \frac{B_2}{2!} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$ □

In fact

$$\zeta(4) = \frac{\pi^4}{90} \qquad \zeta(6) = \frac{\pi^6}{945}$$

and so on.

No simple expression is known for the values $\zeta(2m+1)$ of $\zeta(s)$ at the odd natural numbers. Recall the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s).$$

Here $\Gamma(s)$ is the function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,\mathrm{d}x.$$

One can show that

$$\Gamma(z+1) = z\Gamma(z)$$

so that $\Gamma(n) = (n-1)!$.

It follows that $\zeta(-2n) = 0$, for $n > 0$ and

$$\begin{aligned}\zeta(1-2n) &= 2^{1-2n}\pi^{-2n}\zeta(2n)\Gamma(2n)\sin\left(\frac{\pi}{2} - n\pi\right)\\ &= 2^{1-2n}\pi^{-2n}(-1)^{n-1}\frac{2^{2n}\pi^{2n}}{2(2n)!}B_{2n}(2n-1)!(-1)^n\\ &= -\frac{B_{2n}}{2}.\end{aligned}$$