13. Orders of magnitude

The quantity $\pi(x)$ is relatively easy to approximate with smooth functions. Estimating some of the other quantities which appear in number theory is not so straightforward, as they jump up and down with small changes in n.

Consider $\tau(n)$, the number of divisors of n. We expect this function to grow much slower than n, $\tau(n) = o(n)$. On the one hand, infinitely often $\tau(n)$ is equal to 2, when n is prime and on the other hand, $\tau(n)$ is unbounded from above.

If we plot the points $(n, \tau(n))$ in the plane $\mathbb{R}^2$ note that there is a unique polygonal path with the following properties:

(1) the function $y = T(x)$ whose graph is the polygonal path is piecewise linear,
(2) every point $(n, \tau(n))$ lies below the graph, and
(3) T is convex.

One possible goal is then to give asymptotics for the function $y = T(x)$.

Theorem 13.1.

(1) The relation $\tau(n) \ll \log^h(n)$ does not hold for any h.
(2) $\tau(n) \ll n^\delta$ holds for any $\delta > 0$.

Proof. We first prove (1). Fix r primes $p_1, p_2, \dots, p_r$ and consider

$$n = (2 \cdot 3 \cdot \dots \cdot p_r)^m.$$

Then

$$\begin{aligned} \tau(n) &= \prod_{i=1}^{r}(m+1) \\ &= (m+1)^r \\ &> m^r. \end{aligned}$$

But

$$m = \frac{\log n}{\log(2 \cdot 3 \cdot \dots \cdot p_r)}$$

and so

$$\begin{aligned} \tau(n) &> \frac{\log^r n}{(\log(2 \cdot 3 \cdot \dots \cdot p_r))^r} \\ &\gg \log^r n. \end{aligned}$$

This is (1).

Now we prove (2). If

$$f(n) = \frac{\tau(n)}{n^\delta}$$

then f is multiplicative. We have

$$f(p^m) = \frac{m+1}{p^{m\delta}}.$$

Now p^m tends to infinity if and only if at least one of p or m tends to infinity in which case $f(p^m)$ tends to zero. This is (2). □

In fact we can do a little bit better for (2) of (13.1). As before let $\delta > 0$. Let

$$n = \prod_{i=1}^{r} p_i^{e_i}.$$

We have

$$\begin{aligned} \frac{\tau(n)}{n^\delta} &= \frac{e_1+1}{p_1^{e_1\delta}} \frac{e_2+1}{p_2^{e_2\delta}} \dots \frac{e_n+1}{p_n^{e_n\delta}} \\ &\leq \prod_{p_i \mid n} \max_{x\geq 0} \left(\frac{x+1}{p_i^{\delta x}} \right). \end{aligned}$$

For fixed δ, the quantity

$$\max_{x\geq 0} \left(\frac{x+1}{p^{\delta x}} \right)$$

is equal to 1 for p sufficiently large (for example, consider the Taylor series expansion of e^x: $\log p \geq 1/\delta$ will do) and is never smaller than one.

Thus

$$\frac{\tau(n)}{n^\delta} \leq \prod_{p} \max_{x\geq 0} \left(\frac{x+1}{p^{\delta x}} \right) = c_\delta.$$

Thus we get an explicit constant for (2) of (13.1).

We now turn to Euler's φ-function. We have $\varphi(n) \leq n-1$, with equality if and only if n is prime. We obtained a lower bound in a homework problem. We now establish a better lower bound.

Theorem 13.2.

$$\varphi(n) \gg \frac{n}{\log\log n}.$$

Proof. We have

$$\frac{\varphi(n)}{n} = \prod_{p\mid n} \left(1 - \frac{1}{p}\right).$$

and so taking logs

$$\begin{aligned}\log \frac{\varphi(n)}{n} &= \sum_{p|n} \log\left(1 - \frac{1}{p}\right) \\ &= -\sum_{p|n} \frac{1}{p} + \sum_{p|n} \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \\ &\leq -\sum_{p|n} \frac{1}{p} + O(1),\end{aligned}$$

as

$$\begin{aligned}\sum_{p|n} \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} &> \sum_{p|n} \left(\frac{1}{p} - \frac{1}{p-1}\right) \\ &> -\sum_{k} \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ &= -1.\end{aligned}$$

Let $p_1, p_2, \ldots, p_{r-\rho}$ be the primes less than $\log n$ dividing n and let $p_{r-\rho+1}$, $p_{r-\rho+2}$, $\ldots$, p_r be the remaining primes dividing n,

$$\begin{aligned}\sum_{p|n} \frac{1}{p} &= \sum_{k=1}^{r-\rho} \frac{1}{p_k} + \sum_{k=r-\rho+1}^{r} \frac{1}{p_k} \\ &= S_1 + S_2.\end{aligned}$$

We have

$$\begin{aligned}\log^\rho n &\leq p_{r-\rho+1}^\rho \\ &\leq \prod_{k=r-\rho+1}^{r} p_k \\ &\leq n\end{aligned}$$

and so

$$\rho \leq \frac{\log n}{\log\log n}.$$

Therefore

$$\begin{aligned}S_2 &\leq \frac{1}{\log n} \cdot \frac{\log n}{\log\log n} \\ &= o(1).\end{aligned}$$

By (8.2)

$$\begin{aligned} S_1 &< \log\log p_{r-\rho} + O(1) \\ &< \log\log\log n + O(1). \end{aligned}$$

Putting these two results together we get

$$\log \frac{\varphi(n)}{n} > -\log\log\log n + O(1),$$

so that

$$\frac{\varphi(n)}{n} \gg \frac{1}{\log\log n}. \qquad \square$$

Here is a result that ties $\varphi(n)$ together with $\sigma(n)$.

Theorem 13.3.

$$\frac{1}{2} < \frac{\sigma(n)\varphi(n)}{n^2} < 1.$$

Proof. We have

$$\begin{aligned} \sigma(n)\varphi(n) &= \prod_{p|n} \left(\frac{p^{e+1} - 1}{p - 1} \right) n \prod_{p|n} \left(1 - \frac{1}{p} \right) \\ &= n \prod_{p|n} \left(\frac{1 - p^{-(e+1)}}{1 - 1/p} \right) n \prod_{p|n} \left(1 - \frac{1}{p} \right) \\ &= n^2 \prod_{p|n} \left(1 - p^{-(e+1)} \right). \end{aligned}$$

The coefficient of n^2, the product over the primes, is clearly less than one and at least

$$\begin{aligned} \prod_{p|n} \left(1 - \frac{1}{p^2} \right) &> \prod_{k=1}^{n} \left(1 - \frac{1}{k^2} \right) \\ &> \prod_{k} \left(1 - \frac{1}{k^2} \right) \\ &> \frac{1}{2}, \end{aligned}$$

where the last inequality is shown in (5.3). $\square$

Corollary 13.4.

$$\sigma(n) \ll n \log\log n.$$

Proof. By (13.2) we have

$$\varphi(n) \gg \frac{n}{\log\log n}$$

and so

$$\begin{aligned}\frac{\sigma(n)}{n} &< \frac{n}{\varphi(n)}\\ &\ll \log\log n.\end{aligned}$$

□