12. Bertrands Hypothesis

We will prove:

Theorem 12.1 (Bertrand's hypothesis)**.** *If n is a natural number then there is a prime p such that $n < p \leq 2n$.*

Bertrand's hypothesis was first formulated by Bertrand, who checked it up to six million. He stated it as a hypothesis rather than a conjecture, as we he was very confident it was correct. For example, between $500,000$ and $1,000,000$ there are $36,960$ primes.

(12.1) was proved by Chebyshev. In fact he also proved that for every $\epsilon > 1/5$ there is a number ξ such that for all $x > \xi$ we may find a prime p such that $x \leq p \leq (1+\epsilon)x$ (in fact the prime number theorem implies that this is true for any $\epsilon > 0$).

Lemma 12.2.
$$\prod_{p \leq n} p < 4^n,$$
for every natural number n.

Proof. By induction on n. The cases $n = 1$ and $n = 2$ are clear.

Now suppose it is true up to $n-1$, where $n \geq 3$. If n is even then

$$\begin{aligned}\prod_{p \leq n} p &= \prod_{p \leq n-1} p \\ &< 4^{n-1} \\ &< 4^n.\end{aligned}$$

Thus we may assume that $n = 2m+1$ is odd.

Consider the binomial coefficient
$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}.$$
This is divisible by every prime $m+2 \leq p \leq 2m+1$. Thus
$$\begin{aligned}\prod_{p \leq 2m+1} p &\leq \binom{2m+1}{m} \cdot \prod_{p \leq m+1} p \\ &< \binom{2m+1}{m} \cdot 4^{m+1}.\end{aligned}$$
On the other hand,
$$\binom{2m+1}{m} \qquad \text{and} \qquad \binom{2m+1}{m+1}$$

are equal and both appear in the binomial expansion of $(1+1)^{2m+1}$. Therefore

$$\binom{2m+1}{m} \leq \frac{1}{2} 2^{2m+1} = 4^m.$$

Hence

$$\begin{aligned} \prod_{p \leq 2m+1} p &< 4^m \cdot 4^{m+1} \\ &= 4^{2m+1}. \end{aligned}$$

This completes the induction and the proof. □

Lemma 12.3. *If $n \geq 3$ and $2n/3 < p \leq n$ then p does not divide*

$$\binom{2n}{n}.$$

Proof. Note that

(1) p is greater than 2.
(2) p and $2p$ are the only multiples of p less than or equal to $2n$, since $3p > 2n$.
(3) p is at most n.

(1) and (2) imply $(2n)!$ is divisible by p^2 but not p^3 and (3) implies that p^2 divides $(n!)^2$. Thus p does not divide

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}.$$ □

Proof of (12.1). We can do the cases $n = 1$ and $n = 2$ by hand.

Suppose there is no prime between n and $2n$, where $n \geq 3$. We will bound n from above. (12.3) implies that if p divides

$$\binom{2n}{n}$$

then $p \leq 2n/3$. Suppose that

$$\binom{2n}{n} = p^e m,$$

where m is coprime to p. By the proof of (7.1)

$$\binom{2n}{n} \Bigg| \left(\prod_{p \leq 2n} p^{r_p} \right)$$

where r_p is the unique integer such that

$$p^{r_p} \leq 2n < p^{r_p+1}.$$

Thus $p^e \leq 2n$.

If $e \geq 2$ then $p \leq \sqrt{2n}$. In particular there are at most $\llcorner\sqrt{2n}\lrcorner$ primes in the prime factorisation of

$$\binom{2n}{n}$$

with exponent greater than 1. Therefore

$$\binom{2n}{n} \leq (2n)^{\llcorner\sqrt{2n}\lrcorner} \cdot \prod_{p \leq 2n/3} p.$$

On the other hand

$$\binom{2n}{n}$$

is the largest of the $2n+1$ terms in the expansion of $(1+1)^{2n}$, so that

$$4^n < (2n+1)\binom{2n}{n}.$$

It follows that

$$4^n < (2n+1)(2n)^{\sqrt{2n}} \cdot \prod_{p \leq 2n/3} p.$$

(12.2) implies that

$$4^n < (2n+1)(2n)^{\sqrt{2n}} \cdot 4^{2n/3}.$$

As $2n+1 < 4n^2$ we get

$$4^n < (2n)^{\sqrt{2n}+2} \cdot 4^{2n/3},$$

so that

$$4^{n/3} < (2n)^{\sqrt{2n}+2}.$$

Taking logs gives

$$\frac{n \log 4}{3} < (\sqrt{2n}+2)\log(2n).$$

Note that

$$512 = 2^9.$$

If we plug in $n = 512$ to the equation above the RHS is

$$(2^5+2)10\log 2 = 340\log 2$$

and the LHS is

$$\frac{2^{10}\log 2}{3} > 341\log 2.$$

Thus $n < 512$. Thus if $n \geq 512$ there is a prime between n and $2n$.

On the other hand, for the sequence of primes

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 557,$$

each prime is smaller than twice the preceding prime, so that there is a prime between n and $2n$ for $n < 512$ as well. $\square$