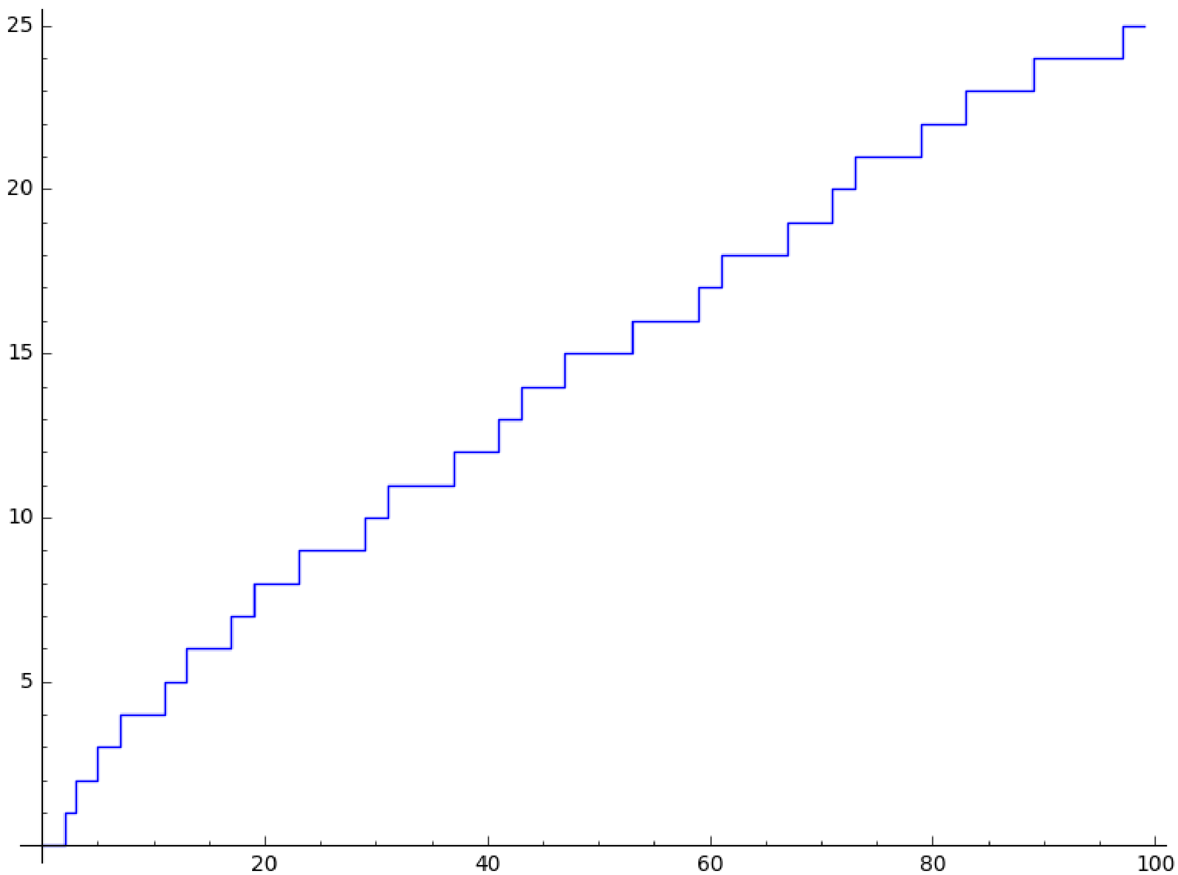
11. Good approximations of $\pi(x)$

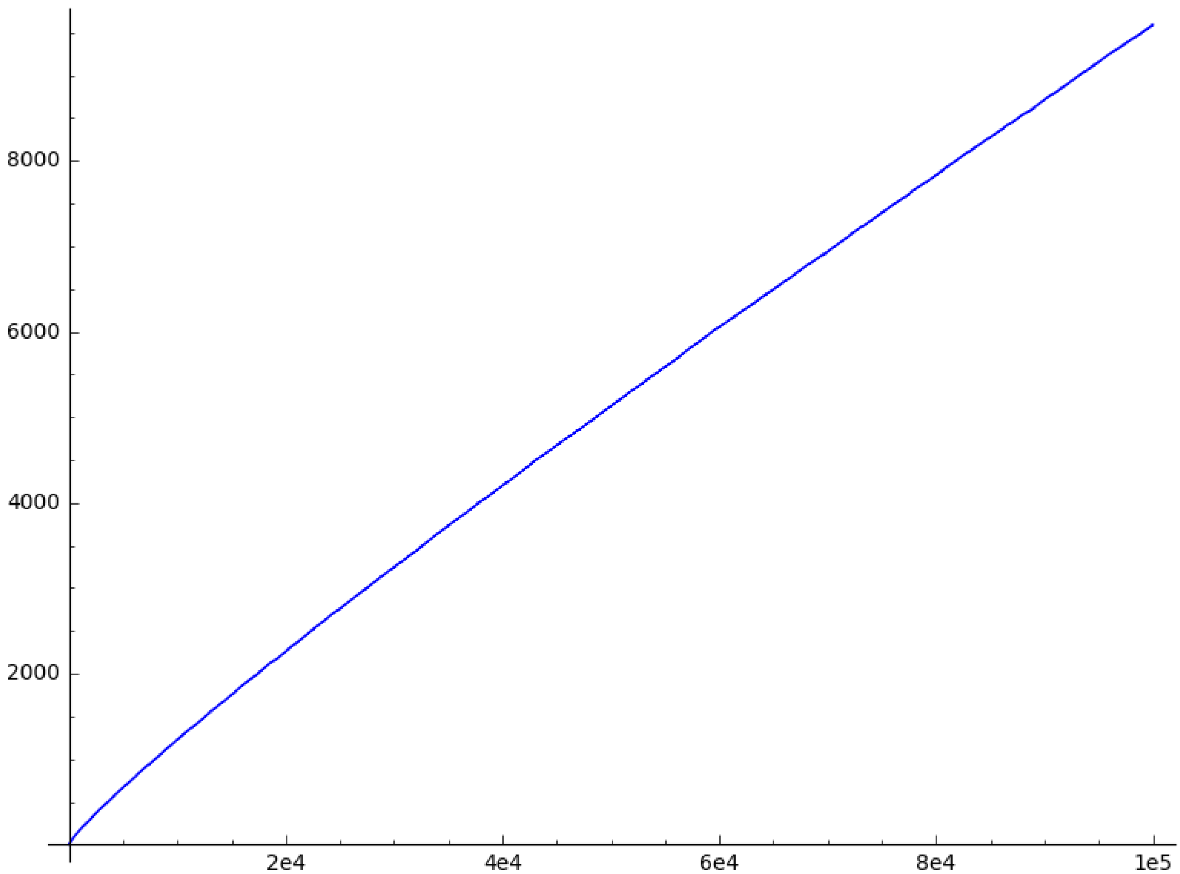
Somewhat paradoxically, primes are distributed in both a random way and in a very controlled way.

For example, the primes between $9,999,900$ and $10,000,000$ are $9,999,901$, $9,999,907$, $9,999,929$, $9,999,931$, $9,999,937$, $9,999,943$, $9,999,971$, $9,999,973$, $9,999,991$. On the other hand, the primes between $10,000,000$ and $10,000,100$ are $10,000,019$ and $10,000,079$. This would seem somewhat random and arbitrary.

It is interesting to plot $\pi(x)$ over larger and larger ranges. At first the graph looks pretty rough. Here is what happens up to $x = 100$.



But if we increase the range to $x = 100,000$ then the graph is unbelievably smooth:



It is interesting to look at tables of $\pi(x)$.

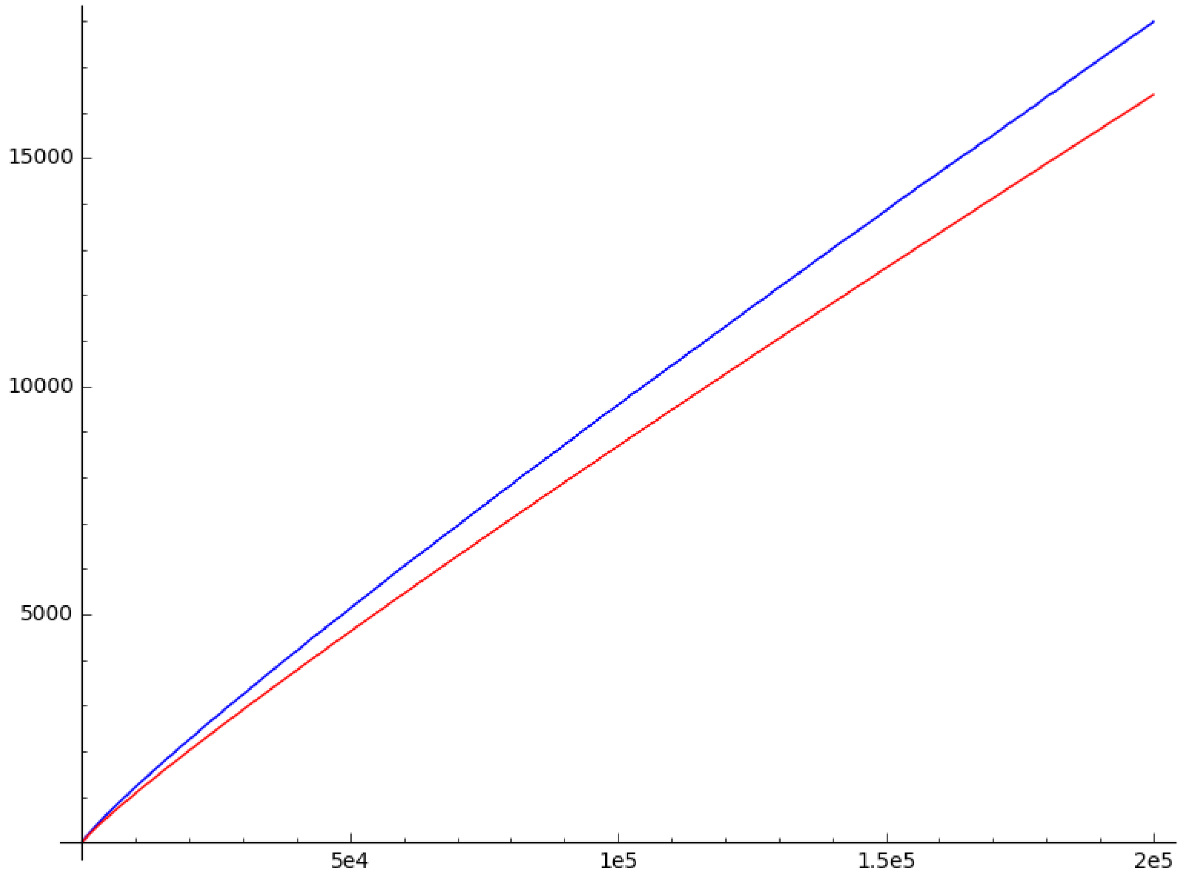
x	$\pi(x)$	$x/\pi(x)$
10	4	2.5
100	25	4.0
1000	168	6.0
10,000	1,229	8.1
100,000	9,592	10.4
1,000,000	78,498	12.7
10,000,000	664,579	15.0
100,000,000	5,761,455	17.4
1,000,000,000	50,847,534	19.7
10,000,000,000	455,052,512	22.0

Note that the ratio between x and $\pi(x)$ jumps up by roughly 2.3. Note that $\log 10 \approx 2.3$. Thus one arrives at a conjecture of the prime number theorem

$$\pi(x) \sim \frac{x}{\log x}.$$

In fact this is how Gauss first conjectured the prime number theorem.

It is interesting to note that if one plots both $\pi(x)$ (in blue) and $x/\log(x)$ (in red) then it is clear there is a lot of room for improvement.



Legendre discovered that a better approximation is given by shifting,

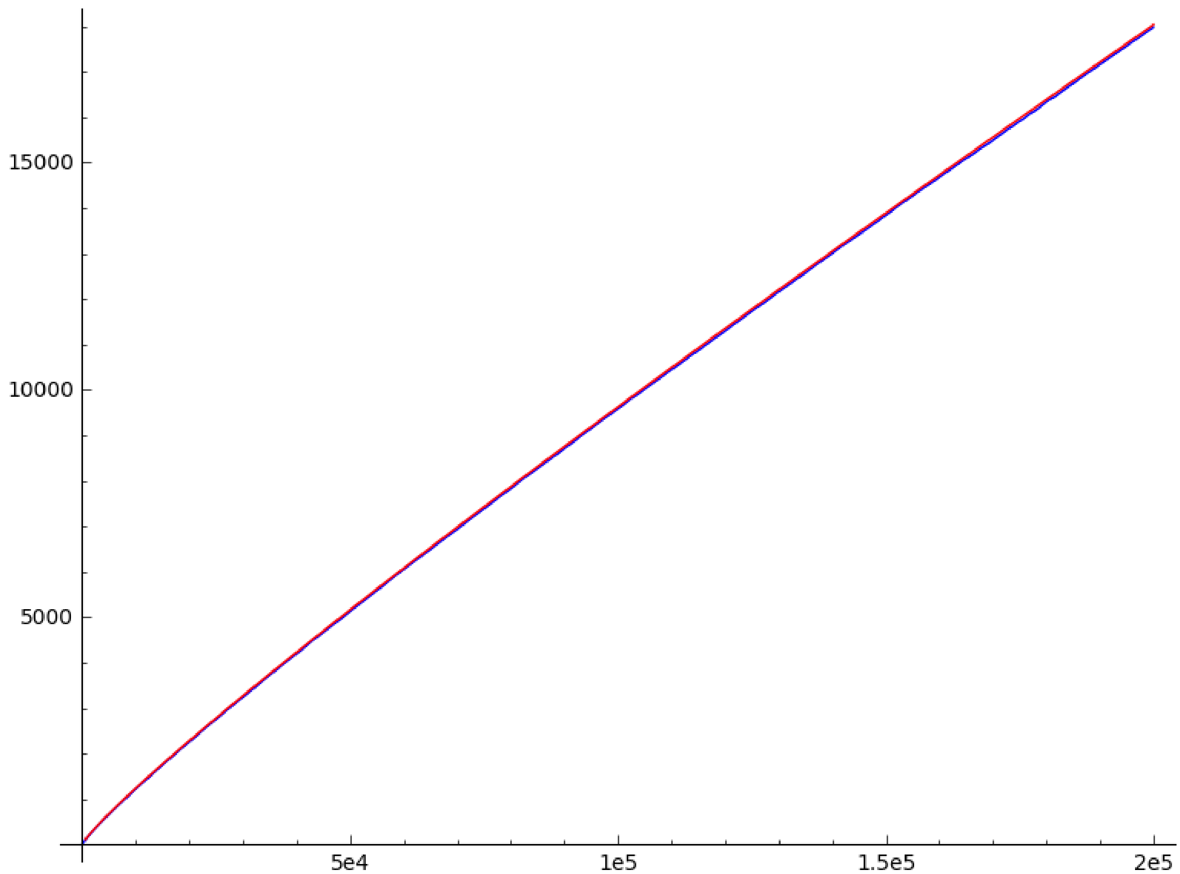
$$\pi(x) \approx \frac{x}{\log x - 1.08366}.$$

Gauss observed that since the frequency of prime numbers at a very large number x is roughly $1/\log x$, a good approximation for $\pi(x)$ ought to be given by the logarithmic sum

$$\mathrm{ls}(x) = \frac{1}{\log 2} + \frac{1}{\log 3} + \cdots + \frac{1}{\log x},$$

or perhaps better by the continuous version, the logarithmic integral:

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log t}\,\mathrm{d}t.$$



Riemann observed that in fact one should expect $1/\log(x)$ to count not just the proportion of primes but the proportion of primes powers, counting the square of a prime as half a prime and so on.

Thus
$$\pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \cdots \approx \mathrm{li}(x).$$
Equivalently
$$\pi(x) \approx \mathrm{li}(x) - \frac{1}{2}\,\mathrm{li}(\sqrt{x}) - \frac{1}{3}\,\mathrm{li}(\sqrt[3]{x}) - \cdots + \frac{\mu(n)}{n}\,\mathrm{li}(x^{1/n}) + \ldots.$$
The function on the RHS is denoted $R(x)$, in honour of Riemann. It gives an amazingly good approximation to $\pi(x)$.

In fact
$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n\zeta(n+1)} \frac{(\log x)^n}{n!}.$$

Note that $\mathrm{li}(x) - R(x)$ is always positive, so that one would expect $\mathrm{li}(x)$ to always be bigger than $\pi(x)$. In fact for most values of x this is true but Littlewood showed the graphs of $\mathrm{li}(x)$ and $\pi(x)$ cross infinitely often and Skewes showed that this happens for some x at most

$$10^{10^{10^{34}}}.$$

In fact Riemann proved that

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho})$$

where the sum ranges over the zeroes of $\zeta(s)$. One can get successively better approximations to the RHS, $R_k(x)$. The graph of $R_{10}(x)$ over the range $x \leq 100$ is a very good approximation to $\pi(x)$ and the graph of $R_{29}(x)$ is an even better approximation.

The prime number theorem is essentially equivalent to the statement that $\zeta(s)$ has no zeroes on the line $\mathrm{Re}(s) = 1$. Riemann established that $\zeta(s)$ satisfies a **functional equation**

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s).$$

Here $\Gamma(s)$ is the function

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} \, \mathrm{d}x.$$

One can show that

$$\Gamma(z+1) = z\Gamma(z)$$

so that $\Gamma(n) = (n-1)!$. Using the functional equation, one can extend the Riemann zeta function to the whole complex plane and that it is zero at -2, -4, ... These are called the **trivial zeroes**.

Conjecture 11.1 (Riemann Hypothesis)**.** *If s is a non-trivial zero of $\zeta(s)$ then* $\mathrm{Re}(s) = 1/2$.