10. The prime number theorem

Here we give a proof of the prime number theorem, using some complex analysis.

Theorem 10.1 (Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\log x}.$$

We need to recall some basic facts from complex analysis.

Definition 10.2. *A function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is called* ***analytic*** *around z_0 if $f(z)$ is represented by a power series centred at z_0,*

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots,$$

valid for $|z - z_0| < r$.

More generally we allow the domain to be an open subset $U \subset \mathbb{C}$. Of course we require the power series to converge inside the circle of radius r centred at z_0. The maximal value of r is called the **radius of convergence**.

Example 10.3. *Consider*

$$\frac{1}{1-z}.$$

This is analytic around 0. Indeed

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

and the radius of convergence is 1. In fact the LHS is not defined when $z = 1$ and this automatically places an upper bound on the radius of convergence.

Consider

$$\frac{1}{1-x} \qquad \text{and} \qquad \frac{1}{1+x^2}$$

Both are real functions with a power series with radius of convergence 1. The first function is not defined at $x = 1$ but the second function is defined at $x = \pm 1$. If we extend these functions to complex functions

$$\frac{1}{1-z} \qquad \text{and} \qquad \frac{1}{1+z^2}$$

then note that the first function is not defined at $z = 1$ and the second function is not defined at $z = \pm i$. $z = 1$ and $z = \pm i$ are called **poles** of the corresponding functions.

Note that we can differentiate a power series, term by term, to define the derivative of an analytic function.

Definition 10.4. *We say that a complex function $f\colon U \longrightarrow \mathbb{C}$ is* ***holomorphic*** *if the derivative exists*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Example 10.5. *The Riemann zeta-function*

$$\zeta(s) = \sum_k \frac{1}{k^s}$$

extends to a holomorphic function for $\operatorname{Re}(s) > 1$.

Indeed, the series converges for $s > 1$ and one can differentiate term by term.

Note that (10.4) is much more restrictive than differentiability of a real function, since we can approach z_0 along any curve. We have the following amazing:

Theorem 10.6. *If $f\colon U \longrightarrow \mathbb{C}$ is a function then f is holomorphic if and only if it is analytic.*

In particular if a complex function has one derivative then it is infinitely differentiable (since one can differentiate a power series as many times as one wants).

The most interesting thing about analytic functions is that one can often extend them around poles.

Theorem 10.7. *The function*

$$\zeta(s) - \frac{1}{s-1}$$

extends to a holomorphic function for all $\operatorname{Re}(s) > 0$.

Proof. For $\operatorname{Re}(s) > 1$ we have

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s}\,\mathrm{d}x.\end{aligned}$$

The last expression converges absolutely for $\operatorname{Re}(s) > 0$ because

$$\begin{aligned}\left|\int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s}\,\mathrm{d}x\right| &= \left|s\int_n^{n+1}\int_n^x \frac{1}{u^s}\,\mathrm{d}u\right| \\ &\leq \max_{n\leq u\leq n+1}\left|\frac{s}{u^{s+1}}\right| \\ &= \frac{|s|}{n^{\operatorname{Re}(s)+1}},\end{aligned}$$

by the mean value theorem. □

We need a little bit of notation.

Definition 10.8. *Let $U \subset \mathbb{C}$ and let $z_0 \in U$ be a point of U. If $f(z)$ is a holomorphic function on $U - \{z_0\}$ and we can write $f(z) = (z-z_0)^\mu g(z)$ where $g(z)$ is holomorphic on U and not zero at z_0 then we say that $f(z)$ has a **zero of order** μ if $\mu \geq 0$ and a **pole of order** $-\mu$ if $\mu < 0$.*

Definition-Lemma 10.9. *Let $f(z)\colon U \longrightarrow \mathbb{C}$ be a holomorphic function.*

*The **logarithmic derivative** of $f(z)$ is equal to*

$$\frac{\mathrm{d}\log f(z)}{\mathrm{d}z}$$

If z_0 is a zero of $f(z)$ of order μ then

$$\frac{f'(z)}{f(z)} - \frac{\mu}{z-z_0}$$

is holomorphic.

Proof. By assumption $f(z) = (z-z_0)^\mu g(z)$, where $g(z)$ is holomorphic and non-zero at z_0, so that

$$\log f(z) = \mu\log(z-z_0) + \log g(z).$$

Therefore

$$\frac{\mathrm{d}\log f(z)}{\mathrm{d}z} = \frac{\mu}{z-z_0} + \frac{g'(z)}{g(z)}.$$

□

We introduce notation for two functions we have already seen implicitly:

$$\Phi(s) = \sum_p \frac{\log p}{p^s} \qquad \text{and} \qquad \vartheta(x) = \sum_{p\leq x}\log p.$$

Theorem 10.10. *If $\operatorname{Re}(s) \geq 1$ then*

- $\zeta(s) \neq 0$
- $\Phi(s) - 1/(s-1)$ *is holomorphic.*

Proof. If $\mathrm{Re}(s) > 1$ then

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

by the usual argument and the RHS is not zero. We use this to compute the logarithmic derivative

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}\log\zeta(s)}{\mathrm{d}s} \\
&= \sum_p \frac{\mathrm{d}\log(1-p^{-s})}{\mathrm{d}s} \\
&= \sum_p \frac{\mathrm{d}\log(1-e^{-s\log p})}{\mathrm{d}s} \\
&= \sum_p \frac{\log p e^{-s\log p}}{1-e^{-s\log p}} \\
&= \sum_p \frac{\log p}{p^s-1} \\
&= \Phi(s) + \frac{\log p}{p^s(p^s-1)}.
\end{aligned}$$

The last sum converges for $\mathrm{Re}(s) > 1/2$. Thus $\Phi(s)$ extends to a holomorphic function away from $s = 1$ and the zeroes of $\zeta(s)$, where it has poles. Suppose that $\zeta(s)$ has a zero of order μ at $1 + i\alpha$ and a zero of order ν at $1 + 2i\alpha$. If we let the real number ϵ go to zero from above then we get

$$\lim_{\epsilon\to 0^+} \epsilon\Phi(1+\epsilon) = 1 \quad \lim_{\epsilon\to 0^+} \epsilon\Phi(1+\epsilon\pm i\alpha) = -\mu \quad \text{and} \quad \lim_{\epsilon\to 0^+} \epsilon\Phi(1+\epsilon\pm 2i\alpha) = -\nu.$$

On the other hand,

$$\begin{aligned}
6 - 8\mu - 2\nu &= \lim_{\epsilon\to 0^+} \sum_{r=-2}^{r=2} \binom{4}{2+r} \Phi(1+\epsilon\pm ri\alpha) \\
&= \lim_{\epsilon\to 0^+} \sum_p \frac{\log p}{p^{1+\epsilon}} (p^{i\alpha/2} + p^{-i\alpha/2})^4 \\
&\geq 0.
\end{aligned}$$

Thus $\mu = 0$ and so $\zeta(s)$ has no zeroes on the line $\mathrm{Re}(s) = 1$. $\square$

Here is the main reason we need complex analysis.

Theorem 10.11. *Let $f(t) \geq 0$ be a bounded and locally integrable function. Suppose that the function*

$$g(z) = \int_0^\infty f(t)e^{-zt}\,\mathrm{d}t$$

defined for $\mathrm{Re}(z) > 0$ *extends holomorphically to* $\mathrm{Re}(z) \geq 0$.

Then

$$\int_0^\infty f(t)\,\mathrm{d}t$$

is a convergent integral (equal to $g(0)$).

Corollary 10.12.

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2}\,\mathrm{d}x$$

is a convergent integral.

Proof. We apply the summation formula to

$$\lambda_n = p_n \qquad c_n = \log p_n \qquad \text{and} \qquad f(x) = x^{-s}.$$

We get

$$\sum_{p \leq x} \frac{\log p}{p^s} = \frac{\vartheta(x)}{x^s} + s\int_1^x \frac{\vartheta(u)}{u^{s+1}}\,\mathrm{d}u.$$

Recall that we already proved that

$$\vartheta(x) = O(x).$$

If $\mathrm{Re}(s) > 1$ then taking the limit as x goes to infinity we get

$$\begin{aligned}\Phi(s) &= \sum_p \frac{\log p}{p^s} \\ &= s\int_1^\infty \frac{\vartheta(u)}{u^{s+1}}\,\mathrm{d}u \\ &= s\int_0^\infty \vartheta(e^t)e^{-st}\,\mathrm{d}t.\end{aligned}$$

Now apply (10.11) to the two functions $f(t) = \vartheta(e^t)e^{-t} - 1$ and

$$g(z) = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}. \qquad \square$$

Theorem 10.13. $\vartheta(x) \sim x$.

Proof. Suppose that $\vartheta(x) \geq \lambda x$ for all x sufficiently large, for some $\lambda > 1$. As $\vartheta(x)$ is monotonic increasing, we have

$$\begin{aligned} \int_x^{\lambda x} \frac{\vartheta(t) - t}{t^2}\,\mathrm{d}t &\geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,\mathrm{d}t \\ &= \int_1^{\lambda} \frac{\lambda - t}{t^2}\,\mathrm{d}t \\ &> 0, \end{aligned}$$

for any such x. If we sum over the intervals $[\lambda x, x]$, $[\lambda x, \lambda^2 x]$ and so on, this contradicts (10.12).

Similarly, now suppose that $\vartheta(x) \leq \lambda x$ for all x sufficiently large, for some $\lambda < 1$. As $\vartheta(x)$ is monotonic increasing, we have

$$\begin{aligned} \int_{\lambda x}^{x} \frac{\vartheta(t) - t}{t^2}\,\mathrm{d}t &\geq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2}\,\mathrm{d}t \\ &= \int_{\lambda}^{1} \frac{\lambda - t}{t^2}\,\mathrm{d}t \\ &< 0, \end{aligned}$$

for any such x. If we sum over the intervals $[\lambda x, x]$, $[x, 1/\lambda x]$ and so on, this contradicts (10.12). ☐

Proof of (10.1). It is a homework problem to show that (10.1) follows from (10.13). ☐

We will need

Theorem 10.14 (Cauchy's integral formula). *Let $f(z)$ be a holomorphic function on a region U and let C be a closed contour that goes once around $z_0 \in U$.*

Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\,\mathrm{d}z.$$

Proof of (10.11). For $T > 0$ let

$$g_T(z) = \int_0^T f(t)\,\mathrm{d}t.$$

Then $g_T(z)$ is holomorphic for all z and we want to show that

$$\lim_{T \to \infty} g_T(0) = g(0).$$

Let C be the boundary of the region

$$\{\, z \in \mathbb{C} \mid |z| \leq R, \mathrm{Re}(z) \geq -\delta \,\}$$

where R is large and δ is small enough (depending on R) so that $g(z)$ is holomorphic on C. Then (10.14) implies that

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{\mathrm{d}z}{z}.$$

Now on the semicircle C_+, the part of the boundary C where the real part is positive, the integrand is bounded by

$$\frac{2B}{R^2},$$

where $B = \max_{t \geq 0} |f(t)|$, since

$$\begin{aligned} |g(z) - g_T(z)| &= \left| \int_T^\infty f(t) e^{-zt} \, \mathrm{d}t \right| \\ &\leq B \int_T^\infty |e^{-zt}| \, \mathrm{d}t \\ &= B \frac{e^{-\operatorname{Re}(z) T}}{\operatorname{Re}(z)}, \end{aligned}$$

and

$$\left| e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} \right| = e^{\operatorname{Re}(z) T} \frac{2 \operatorname{Re}(z)}{R^2}.$$

As the semi-circle C_+ has length πR the absolute value of the integral over C_+ is bounded by B/R.

Now consider the integral over C_-, where the real part is negative. We deal with $g(z)$ and $g_T(z)$ separately. Since $g_T(z)$ is entire, that is, everywhere holomorphic, we can replace C_- by the semi-circle C'_-, the other half of C_+. We get the same bound as before, since

$$\begin{aligned} |g_T(z)| &= \left| \int_0^T f(t) e^{-zt} \, \mathrm{d}t \right| \\ &\leq B \int_{-\infty}^T |e^{-zt}| \, \mathrm{d}t \\ &= B \frac{e^{-\operatorname{Re}(z) T}}{|\operatorname{Re}(z)|}. \end{aligned}$$

Thus the integral of $g_T(z)$ over C_- goes to zero as R goes to infinity.

Finally the remaining integral over C_- also goes to zero as R goes to infinity, as the integrand is the product of

$$\frac{g(z)}{z} \left(1 + \frac{z^2}{R^2}\right),$$

which is independent of T and the function e^{zT}, which goes to zero rapidly as T goes to infinity.

Thus

$$\limsup_{T \to \infty} |g(0) - g_T(0)| \leq \frac{2B}{R}.$$

Now let R go to infinity. $\square$