

1. MULTIPLICATIVE FUNCTIONS

The focus of Math 104B will be on giving upper and lower bounds for functions defined on the natural numbers. In Math 104A the focus was on using algebra to solve problems in number theory. By contrast in Math 104B we will use analysis instead.

We recall the definition of two closely related functions

Definition 1.1. *We define two functions*

$$\tau: \mathbb{N} \longrightarrow \mathbb{N} \quad \text{and} \quad \sigma: \mathbb{N} \longrightarrow \mathbb{N}$$

as follows: if $n \in \mathbb{N}$ then $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is the sum of the divisors of n .

We can write down the definitions more formally:

$$\tau(n) = \sum_{d|n} 1 \quad \text{and} \quad \sigma(n) = \sum_{d|n} d.$$

For example, consider $n = 6$. The divisors of 6 are 1, 2, 3, and 6. Thus

$$\tau(6) = 4 \quad \text{and} \quad \sigma(6) = 1 + 2 + 3 + 6 = 12.$$

We recall the definition of a multiplicative function:

Definition 1.2. *We say that a function*

$$f: \mathbb{N} \longrightarrow \mathbb{N}$$

*is **multiplicative** if $f(mn) = f(m)f(n)$, whenever m and n coprime.*

To compute a multiplicative function f , by the fundamental theorem of arithmetic, it suffices to know the value of $f(p^e)$, where p is a prime number.

We have already seen that the Euler φ -function is multiplicative.

Theorem 1.3. *The functions σ and τ are multiplicative.*

Proof. Suppose that m and n are two coprime natural numbers. Then every divisor d of mn is uniquely of the form d_1d_2 , where d_1 divides m and d_2 divides n .

It follows that the number of divisors of mn is equal to the number of divisors of m multiplied by the number of divisors of n , that is, $\tau(mn) = \tau(m)\tau(n)$. In particular τ is multiplicative.

On the other hand,

$$\begin{aligned}
 \sigma(m)\sigma(n) &= \left(\sum_{d_1|m} d_1 \right) \left(\sum_{d_2|n} d_2 \right) \\
 &= \sum_{d_1|m, d_2|n} d_1 d_2 \\
 &= \sum_{d|mn} d \\
 &= \sigma(mn). \quad \square
 \end{aligned}$$

Lemma 1.4. *If p is a prime and e is a natural number then*

$$\tau(p^e) = 1 + e \quad \text{and} \quad \sigma(p^e) = \frac{p^{e+1} - 1}{p - 1}.$$

Proof. The divisors of p^e are $1, p, p^2, \dots, p^{e-1}, p^e$. The number of divisors is then $1 + e$ so that

$$\tau(p^e) = 1 + e$$

and the sum of the divisors is

$$\begin{aligned}
 \sigma(p^e) &= 1 + p + p^2 + p^3 + \dots + p^{e-1} + p^e \\
 &= \frac{p^{e+1} - 1}{p - 1}. \quad \square
 \end{aligned}$$

Theorem 1.5. *If*

$$n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$$

is the prime factorisation of the natural number n then

$$\tau(n) = \prod_{i=1}^r (1 + e_i) \quad \text{and} \quad \sigma(n) = \prod_{i=1}^r \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Proof. Immediate from (1.3) and (1.4). □

In fact there is a way to generate lots of multiplicative functions of which σ and τ are special cases.

Theorem 1.6. *If $f: \mathbb{N} \rightarrow \mathbb{N}$ is a multiplicative function then the function $F: \mathbb{N} \rightarrow \mathbb{N}$ defined by*

$$F(n) = \sum_{d|n} f(d)$$

is also multiplicative.

Proof. Suppose that m and n are two coprime natural numbers. Then every divisor d of mn is uniquely of the form d_1d_2 , where d_1 divides m and d_2 divides n .

Therefore

$$\begin{aligned}
 F(m)F(n) &= \left(\sum_{d_1|m} f(d_1) \right) \left(\sum_{d_2|n} f(d_2) \right) \\
 &= \sum_{d_1|m, d_2|n} f(d_1)f(d_2) \\
 &= \sum_{d_1|m, d_2|n} f(d_1d_2) \\
 &= \sum_{d|mn} f(d) \\
 &= F(mn). \quad \square
 \end{aligned}$$

Corollary 1.7. σ and τ are multiplicative functions.

Proof. Apply (1.6) to the functions $f(n) = 1$ and $f(n) = n$, which are both easily seen to be multiplicative. \square

Definition 1.8. A natural number n is called **perfect** if the sum of the divisors, apart from n , is equal to n .

Thus 6 is perfect as $6 = 1 + 2 + 3$. Note that n is perfect if and only if $\sigma(n) = 2n$.

Theorem 1.9. Let n be an even natural number.

Then n is perfect if and only if $n = 2^{p-1}(2^p - 1)$ where both p and $2^p - 1$ are prime numbers.

Proof. One direction is straightforward. If $n = 2^{p-1}(2^p - 1)$ where both p and $2^p - 1$ are prime then

$$\begin{aligned}
 \sigma(n) &= \sigma(2^{p-1}(2^p - 1)) \\
 &= \sigma(2^{p-1})\sigma(2^p - 1) \\
 &= \frac{2^p - 1}{2 - 1} 2^p \\
 &= 2^p(2^p - 1) \\
 &= 2n,
 \end{aligned}$$

so that n is perfect.

Now suppose that n is perfect. We may write $n = 2^{k-1}m$, where m is odd and $k \geq 2$ as n is even. We have

$$\begin{aligned} 2n &= \sigma(n) \\ &= \sigma(2^{k-1}m) \\ &= \sigma(2^{k-1})\sigma(m) \\ &= (2^k - 1)\sigma(m). \end{aligned}$$

Thus

$$(2^k - 1)\sigma(m) = 2^k m.$$

In particular $(2^k - 1) \mid m$. Thus we may write $m = (2^k - 1)l$, for some natural number l , in which case

$$\sigma(m) = 2^k l.$$

Now both l and m are divisors of m so that

$$\begin{aligned} 2^k l &= \sigma(m) \\ &\geq m + l \\ &= (2^k - 1)l + l \\ &= 2^k l. \end{aligned}$$

It follows that l and m are the only divisors of m , so that m is prime and $l = 1$. It follows that $m = 2^k - 1$ is prime, which only happens if k is prime. \square

There are two natural questions about perfect numbers.

Question 1.10. *Are there odd perfect numbers?*

It has been checked by computer that there are no odd perfect numbers less than 10^{300} ; we know that if n is an odd perfect number not divisible by 3, 5 or 7 then n is divisible by at least 27 different primes; every odd perfect number is congruent to 1 modulo 12 or 9 modulo 36. Presumably there are no odd primes but the answer to (1.10) is not known.

Question 1.11. *Are there infinitely many perfect numbers?*

If we assume that there are only finitely many odd perfect numbers (for example, if there are none) this is equivalent to asking if there are infinitely many primes p such that

$$2^p - 1,$$

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is prime. Recall that any such prime is called a **Mersenne prime**. To date we know the existence of 50 Mersenne primes. The primes

3, 7, 31, and 127.

are the first four Mersenne primes. The 50th Mersenne prime is

$$2^{77,232,917} - 1.$$

It has 23,249,425 digits, and is the largest known prime number. It was discovered on Dec 26th, 2017.