MODEL ANSWERS TO THE NINTH HOMEWORK

9.3.1. $\sqrt{2} = 1 + \sqrt{2} - 1$, so that $a_0 = 1$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{2}-1}\\ &= \sqrt{2}+1\\ &= 2+\sqrt{2}-1,\end{aligned}$$

so that $a_1 = 2$. It follows that

$$\sqrt{2} = [1; \overline{2}].$$

$\sqrt{3} = 1 + \sqrt{3} - 1$, so that $a_0 = 1$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{3}-1}\\ &= \frac{\sqrt{3}+1}{2}\\ &= 1+\frac{\sqrt{3}-1}{2}.\end{aligned}$$

It follows that $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \frac{2}{\sqrt{3}-1}\\ &= \sqrt{3}+1\\ &= 2+\sqrt{3}-1.\end{aligned}$$

It follows that

$$\sqrt{3} = [1; \overline{1, 2}].$$

$\sqrt{5} = 2 + \sqrt{5} - 2$, so that $a_0 = 2$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{5}-2}\\ &= \sqrt{5}+2\\ &= 4+\sqrt{5}-2.\end{aligned}$$

It follows that

$$\sqrt{5} = [2; \overline{4}].$$

$\sqrt{6} = 2 + \sqrt{6} - 2$, so that $a_0 = 2$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{6}-2} \\ &= \frac{\sqrt{6}+2}{2} \\ &= 2 + \frac{\sqrt{6}-2}{2}.\end{aligned}$$

It follows that $a_1 = 2$ and

$$\begin{aligned}\xi_2 &= \frac{2}{\sqrt{6}-2} \\ &= \sqrt{6}+2 \\ &= 4 + \sqrt{6} - 2.\end{aligned}$$

It follows that

$$\sqrt{6} = [2; \overline{2, 4}].$$

$\sqrt{7} = 2 + \sqrt{7} - 2$, so that $a_0 = 2$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{7}-2} \\ &= \frac{\sqrt{7}+2}{3} \\ &= 1 + \frac{\sqrt{7}-1}{3}.\end{aligned}$$

It follows that $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \frac{3}{\sqrt{7}-1} \\ &= \frac{\sqrt{7}+1}{2} \\ &= 1 + \frac{\sqrt{7}-1}{2}.\end{aligned}$$

It follows that $a_2 = 1$ and

$$\begin{aligned}\xi_3 &= \frac{2}{\sqrt{7}-1} \\ &= \frac{\sqrt{7}+1}{3} \\ &= 1 + \frac{\sqrt{7}-2}{3}.\end{aligned}$$

It follows that $a_3 = 1$ and

$$\begin{aligned}\xi_3 &= \frac{3}{\sqrt{7}-2} \\ &= \sqrt{7}+2 \\ &= 4+\sqrt{7}-2.\end{aligned}$$

It follows that

$$\sqrt{7} = [2; \overline{1,1,1,4}].$$

$\sqrt{8} = 2\sqrt{2} = 2 + 2\sqrt{2} - 2$, so that $a_0 = 2$ and

$$\begin{aligned}\xi_1 &= \frac{1}{2(\sqrt{2}-1)} \\ &= \frac{\sqrt{2}+1}{2} \\ &= 1 + \frac{(\sqrt{2}-1)}{2}.\end{aligned}$$

It follows that $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \frac{2}{(\sqrt{2}-1)} \\ &= 2(\sqrt{2}+1) \\ &= 4 + 2(\sqrt{2}-1).\end{aligned}$$

It follows that

$$\sqrt{8} = [2; \overline{1,4}].$$

$\sqrt{10} = 3 + (\sqrt{10} - 3)$, so that $a_0 = 3$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{10}-3} \\ &= \sqrt{10}+3 \\ &= 6+\sqrt{10}-3.\end{aligned}$$

It follows that

$$\sqrt{10} = [3; \overline{6}]$$

$\sqrt{11} = 3 + (\sqrt{11} - 3)$, so that $a_0 = 3$

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{11}-3} \\ &= \frac{\sqrt{11}+3}{2} \\ &= 3 + \frac{\sqrt{11}-3}{2},\end{aligned}$$

It follows that $a_1 = 3$ and

$$\begin{aligned}\xi_2 &= \frac{2}{\sqrt{11}-3} \\ &= \sqrt{11}+3 \\ &= 6 + \sqrt{11} - 3.\end{aligned}$$

It follows that

$$\sqrt{11} = [3; \overline{3, 6}].$$

$\sqrt{12} = 3 + (2\sqrt{3} - 3)$, so that $a_0 = 3$ and

$$\begin{aligned}\xi_1 &= \frac{1}{2\sqrt{3}-3} \\ &= \frac{2\sqrt{3}+3}{3} \\ &= 2 + \frac{2\sqrt{3}-3}{3}.\end{aligned}$$

It follows that $a_1 = 2$ and

$$\begin{aligned}\xi_2 &= \frac{3}{2\sqrt{3}-3} \\ &= 2\sqrt{3}+3 \\ &= 6 + 2\sqrt{3} - 3.\end{aligned}$$

It follows that

$$\sqrt{12} = [3; \overline{2, 6}].$$

$\sqrt{13} = 3 + (\sqrt{13} - 3)$, so that $a_0 = 3$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{13}-3} \\ &= \frac{\sqrt{13}+3}{4} \\ &= 1 + \frac{\sqrt{13}-1}{4}.\end{aligned}$$

It follows that $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \frac{4}{\sqrt{13}-1} \\ &= \frac{\sqrt{13}+1}{3} \\ &= 1 + \frac{\sqrt{13}-2}{3}.\end{aligned}$$

It follows that $a_2 = 1$ and

$$\begin{aligned}\xi_3 &= \frac{3}{\sqrt{13}-2} \\ &= \frac{\sqrt{13}+2}{3} \\ &= 1 + \frac{\sqrt{13}-1}{3}.\end{aligned}$$

It follows that $a_3 = 1$ and

$$\begin{aligned}\xi_4 &= \frac{3}{\sqrt{13}-1} \\ &= \frac{\sqrt{13}+1}{4} \\ &= 1 + \frac{\sqrt{13}-3}{4}.\end{aligned}$$

It follows that $a_4 = 1$ and

$$\begin{aligned}\xi_5 &= \frac{4}{\sqrt{13}-3} \\ &= \sqrt{13}+3 \\ &= 6 + \sqrt{13}-3.\end{aligned}$$

It follows that

$$\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}].$$

$\sqrt{14} = 3 + (\sqrt{14} - 3)$, so that $a_0 = 3$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{14}-3} \\ &= \frac{\sqrt{14}+3}{5} \\ &= 1 + \frac{\sqrt{14}-2}{5}.\end{aligned}$$

It follows that $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \frac{5}{\sqrt{14}-2}\\ &= \frac{\sqrt{14}+2}{2}\\ &= 2 + \frac{\sqrt{14}-2}{2}.\end{aligned}$$

It follows that $a_2 = 2$ and

$$\begin{aligned}\xi_3 &= \frac{2}{\sqrt{14}-2}\\ &= \frac{\sqrt{14}+2}{5}\\ &= 1 + \frac{\sqrt{14}-3}{5}.\end{aligned}$$

It follows that $a_3 = 1$ and

$$\begin{aligned}\xi_3 &= \frac{5}{\sqrt{14}-3}\\ &= \sqrt{14}+3\\ &= 6 + \sqrt{14}-3.\end{aligned}$$

It follows that

$$\sqrt{14} = [3; \overline{1, 2, 1, 6}].$$

$\sqrt{15} = 3 + (\sqrt{15} - 3)$, so that $a_0 = 3$ and

$$\begin{aligned}\xi_1 &= \frac{1}{\sqrt{15}-3}\\ &= \frac{\sqrt{15}+3}{6}\\ &= 1 + \frac{\sqrt{15}-3}{6}.\end{aligned}$$

It follows that $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \frac{6}{\sqrt{15}-3}\\ &= \sqrt{15}+3\\ &= 6 + \sqrt{15}-3.\end{aligned}$$

It follows that

$$\sqrt{15} = [3; \overline{1, 6}].$$

9.3.2. As $\xi < \theta < \eta$, it follows that

$$a_0 = \llcorner \xi \lrcorner \leq \llcorner \theta \lrcorner \leq \llcorner \eta \lrcorner = a_0.$$

Thus

$$a_0 = \llcorner \theta \lrcorner.$$

Moreover, it then follows that

$$\{\,\xi\,\} < \{\,\theta\,\} < \{\,\eta\,\}.$$

Taking reciprocals

$$\eta_1 < \theta_1 < \xi_1.$$

As the partial quotients of η_1 and ξ_1 are $a_1, a_2, \ldots, a_n$, we are done by induction on n.

9.3.3. We proceed by induction on n. $q_1 = a_1$ and so we omit no pairs in this case. Suppose the result holds for all $k < n$, where $n > 1$. We have

$$q_n = a_n q_{n-1} + q_{n-2}.$$

q_{n-2} is the product of $a_1 a_2 \ldots a_n$, omitting some pairs of consecutive integers of $a_1 a_2 \ldots a_{n-2}$, including $a_{n-1} a_{n-2}$. $a_n q_{n-1}$ is the product of $a_1 a_2 \ldots a_n$ omitting some pairs of consecutive integers of $a_1 a_2 \ldots a_{n-1}$, but never omitting a_n. This completes the induction.

9.3.4.

$$\begin{aligned} u &= [2; 1, 2, 1, 1, 4, 1, 1] \\ &= [2; 1, 2, 1, 1, 4, 2] \\ &= [2; 1, 2, 1, 1, 9/2] \\ &= [2; 1, 2, 1, 11/9] \\ &= [2; 1, 2, 20/11] \\ &= [2; 1, 51/20] \\ &= [2; 71/51] \\ &= \frac{193}{71} \\ &= 2.71830985915. \end{aligned}$$

$$\begin{aligned} l &= [2; 1, 2, 1, 1, 4, 1, 1, 1] \\ &= [2; 1, 2, 1, 1, 4, 1, 2] \\ &= [2; 1, 2, 1, 1, 4, 3/2] \\ &= [2; 1, 2, 1, 1, 14/3] \\ &= [2; 1, 2, 1, 17/14] \\ &= [2; 1, 2, 31/17] \\ &= [2; 1, 79/31] \\ &= [2; 110/79] \\ &= \frac{299}{110} \\ &= 2.7\overline{18}. \end{aligned}$$

The general real number ξ will have a continued fraction which starts like

$$[2; 1, 2, 1, 1, 4, 1, 1]$$

if and only if

$$\xi \in [2.7\overline{18}, 2.71830985915].$$

9.3.5. There is no harm in assuming that $\delta < 1/2$. In particular if

$$\left|\xi - \frac{p}{q}\right| < \frac{\delta}{q^2}$$

then $p/q = p_k/q_k$ is a convergent of the continued fraction expansion of ξ. In this case

$$\frac{1}{q_k(q_k + q_{k+1})} < \left|\xi - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}}.$$

As

$$q_{k+1} = a_{k+1} q_k + q_{k-1}$$

It follows that

$$a_{k+1} q < q^2 < (a_{k+1} + 1) q^2.$$

If $a_k < A$ for some A then

$$\left|\xi - \frac{p}{q}\right| > \frac{\delta}{q^2}$$

for

$$\delta = \frac{1}{A+1}.$$

Conversely if
$$\left|\xi - \frac{p}{q}\right| > \frac{\delta}{q^2}$$
then
$$a_k < \frac{1}{\delta},$$
is bounded.
9.3.7. If $x_{k+1} > q_k^\omega$ then $a_{k+1} \geq q_k^\omega$ and so
$$q_{k+1} = a_{k+1}q_k + q_{k-1} > q_k^{\omega+1}.$$
In this case
$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_k^{\omega+2}}.$$
Therefore x is Liouville number.
If x is a Liouville number and
$$\left|x - \frac{p}{q}\right| < \frac{1}{q^n}$$
then p/q has to be a convergent. It follows that
$$q_k(q_k + q_{k+1}) > q_k^n$$
But then
$$q_{k+1} > \frac{1}{2}q_k^n.$$
9.4.2. (a) We have
$$\begin{aligned} 0 &< \xi - \bar{\xi} \\ &= 2\frac{\sqrt{d}}{B}. \end{aligned}$$
Therefore $B > 0$. On the other hand
$$\begin{aligned} 0 &< \xi + \bar{\xi} \\ &= 2\frac{A}{B}. \end{aligned}$$
Therefore $A > 0$.
As $\xi > 1$ we have
$$\sqrt{d} + A > B \qquad \text{so that} \qquad B < \sqrt{d} + A$$
As $-1 < \bar{\xi} < 0$ we have
$$-B < A - \sqrt{d} < 0 \qquad \text{so that} \qquad \sqrt{d} - B < A < \sqrt{d}.$$
Putting all of this together gives
$$0 < A < \sqrt{d} \qquad \text{and} \qquad 0 < \sqrt{d} - A < B < \sqrt{d} + A < 2\sqrt{d}.$$

(b) As $B < \sqrt{d} + A$ it follows that $\xi > 1$. As

$$\sqrt{d} - B < A < \sqrt{d} \qquad \text{so that} \qquad -B < A - \sqrt{d} < 0.$$

and so $-1 < \bar{\xi} < 0$.
(c) We already now that $\sqrt{d} + \lfloor \sqrt{d} \rfloor$ is reduced and so there is at least one. As $0 < A < \sqrt{d}$ it follows that there are only finitely many possible choices for A. As $0 < B < 2\sqrt{d}$ it follows that there are only finitely many possible choices for B.
9.5.1.

$$\sqrt{7} = [2; \overline{1, 1, 1, 4}].$$

The convergents are

$$\frac{2}{1} \quad \frac{3}{1} \quad \frac{5}{2} \quad \frac{8}{3} \quad \frac{37}{14} \quad \frac{45}{17} \quad \frac{82}{31} \quad \frac{127}{48} \quad \frac{590}{223} \quad \frac{717}{271}$$

If we compute $p_k^2 - 7q_k^2$ we get

$$2^2 - 7 = -3, \quad 3^2 - 7 = 3, \quad 5^2 - 4 \cdot 7 = -3, \quad 8^2 - 9 \cdot 7 = 1, \quad 37^2 - 14 \cdot 7 = -3,$$

and

$$(45)^2 - (17)^2 \cdot 7 = 2, \quad (82)^2 - 31^2 \cdot 7 = -3, \quad (127)^2 - (48)^2 \cdot 7 = 1, \quad (590)^2 - (223)^2 \cdot 7 = -3.$$

Finally we get

$$(717)^2 - (271) \cdot 7 = 2.$$

It follows that we can solve

$$x^2 - 7y^2 = N,$$

where $|N| \leq 2$, if and only if $N = 1$ or $N = 2$.
9.5.2. We find the continued fraction expansion of $\sqrt{95}$. $\sqrt{95} = 9 + \sqrt{95} - 9$, so that $a_0 = 9$. It follows that

$$\begin{aligned} \xi_1 &= \frac{1}{\sqrt{95} - 9} \\ &= \frac{\sqrt{95} + 9}{14} \\ &= 1 + \frac{\sqrt{95} - 5}{14}. \end{aligned}$$

It follows that $a_1 = 1$ and

$$\xi_2 = \frac{14}{\sqrt{95}-5} = \frac{\sqrt{95}+5}{5} = 2 + \frac{\sqrt{95}-5}{5}.$$

It follows that $a_2 = 2$ and

$$\xi_3 = \frac{5}{\sqrt{95}-5} = \frac{\sqrt{95}+5}{14} = 1 + \frac{\sqrt{95}-9}{14}.$$

It follows that $a_3 = 1$ and

$$\xi_4 = \frac{14}{\sqrt{95}-9} = \sqrt{95}+9 = 18 + \sqrt{95}-9.$$

It follows that

$$\sqrt{95} = [9; \overline{1, 2, 1, 18}].$$

The partial quotients are

$$\frac{9}{1} \qquad \frac{10}{1} \qquad \frac{29}{3} \qquad \frac{39}{4}.$$

If we compute $p_k^2 - 95q_k^2$ we get

$$81-95 = -14 \qquad 100-95 = 5 \qquad (29)^2-9\cdot 95 = -14 \qquad (39)^2-16\cdot 95 = 1.$$

Thus the fundamental solution is $39 + 4\sqrt{95}$.
We find the continued fraction expansion of $\sqrt{74}$. $\sqrt{74} = 8 + \sqrt{74} - 8$, so that $a_0 = 8$. It follows that

$$\xi_1 = \frac{1}{\sqrt{74}-8} = \frac{\sqrt{74}+8}{10} = 1 + \frac{\sqrt{74}-2}{10}.$$

It follows that $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \frac{10}{\sqrt{74}-2} \\ &= \frac{\sqrt{74}+2}{7} \\ &= 1 + \frac{\sqrt{74}-5}{7}.\end{aligned}$$

It follows that $a_2 = 1$ and

$$\begin{aligned}\xi_3 &= \frac{7}{\sqrt{74}-5} \\ &= \frac{\sqrt{74}+5}{7} \\ &= 1 + \frac{\sqrt{74}-2}{7}.\end{aligned}$$

It follows that $a_3 = 1$ and

$$\begin{aligned}\xi_4 &= \frac{7}{\sqrt{74}-2} \\ &= \frac{\sqrt{74}+2}{10} \\ &= 1 + \frac{\sqrt{74}-8}{10}.\end{aligned}$$

It follows that $a_4 = 1$ and

$$\begin{aligned}\xi_5 &= \frac{10}{\sqrt{74}-8} \\ &= \sqrt{74}+8 \\ &= 16 + \sqrt{74} - 8.\end{aligned}$$

It follows that

$$\sqrt{74} = [8; \overline{1,1,1,1,16}].$$

The partial quotients are

$$\frac{8}{1} \qquad \frac{9}{1} \qquad \frac{17}{2} \qquad \frac{26}{3} \qquad \frac{43}{5}.$$

If we compute $p_k^2 - 74q_k^2$ we get

$$64-74 = -10, \quad 81-74 = 7, \quad (17)^2-4\cdot 74 = -7, \quad (26)^2-9\cdot 74 = 10, \quad (43)^2-25\cdot 74 = -1.$$

Thus $43 + 5\sqrt{74}$ is the minimal solution of $x^2 - 74y^2 = -1$. Squaring gives the fundamental solution:

$$(43)^2 + 25 \cdot 74 + 2 \cdot 43 \cdot 5\sqrt{74} = 3699 + 430\sqrt{74}.$$