MODEL ANSWERS TO THE EIGHTH HOMEWORK

9.1.1.

$$\begin{aligned}\Phi(t,n) &= \sum_{m=1}^{n} \sum_{1\leq a<tm,(a,m)=1} 1 \\ &= \sum_{m=1}^{n} \varphi(t,m) \\ &= \sum_{m=1}^{n} t\varphi(m) + O(\tau(m)) \\ &= t\Phi(1,n) + O(n\log n).\end{aligned}$$

As

$$\Phi(1,n) = \frac{3n^2}{\pi^2} + O(n\log n)$$

it follows that

$$\Phi(t,n) \sim t\Phi(1,n).$$

9.1.2. We have

$$\begin{aligned}\lambda(c) &= \sum_{p/q\in[0,1]} l(J_c(p/q)) \\ &= \sum_{q=1}^{\infty} \sum_{\substack{0<p\leq q,\\ (p,q)=1}} l(J_c(p/q)) \\ &= \sum_{q=1}^{\infty} \sum_{\substack{0<p\leq q,\\ (p,q)=1}} \frac{2c}{q^\nu} \\ &< \sum_{q=1}^{\infty} \frac{2c}{q^\nu} \sum_{0<p\leq q} 1 \\ &= 2c\sum_{q=1}^{\infty} \frac{1}{q^{\nu-1}},\end{aligned}$$

which converges if $\nu - 1 > 1$ and

$$\lim_{c\to 0} \lambda(c) = 0.$$

9.1.3. Suppose that

$$p^2 - pq - q^2 = 0.$$

and yet $pq \neq 0$. We are going to derive a contradiction. There is no harm in assuming that p and q are coprime. We may write

$$(p-q)(p+q) = pq.$$

If a prime divides p it must divide one of $p+q$ and $p-q$. But then this prime must divide q, a contradiction. Thus p and q are both ± 1. In this case the LHS is zero but not the RHS, a contradiction.
Thus

$$|p^2 - pq + q^2| \geq 1,$$

if not both p and q are zero.
Let

$$\phi = \frac{1+\sqrt{5}}{2},$$

the Golden ratio. Fix $C > 0$, such that

$$C > \sqrt{5}.$$

Suppose that

$$\phi - \frac{p}{q} = \frac{\delta}{q^2},$$

for some

$$|\delta| < \frac{1}{C}.$$

Multiplying by q we get

$$\frac{\delta}{q} = q\phi - p.$$

This gives

$$\frac{\delta}{q} - \frac{q\sqrt{5}}{2} = \frac{q}{2} - p.$$

Squaring both sides and subtracting

$$\frac{5q^2}{4}$$

gives

$$\frac{\delta^2}{q^2} - \delta\sqrt{5} = p^2 - pq - q^2.$$

The first term on the LHS tends to zero as q tends to infinity. As the second term is bigger than -1 and the RHS has magnitude at least one, it follows that there are only finitely many possible values for q. It follows that there are finitely many possible choices for p/q.

9.2.1. $\llcorner\sqrt{3}\lrcorner = 1$ so that $a_0 = 1$ and

$$\begin{aligned}\xi_1 &= (\sqrt{3}-1)^{-1}\\ &= \frac{\sqrt{3}+1}{2}\\ &= 1 + \frac{\sqrt{3}-1}{2}.\end{aligned}$$

Thus $a_1 = 1$ and

$$\begin{aligned}\xi_2 &= \left(\frac{\sqrt{3}-1}{2}\right)^{-1}\\ &= \frac{2}{\sqrt{3}-1}\\ &= \sqrt{3}+1\\ &= 2 + \sqrt{3} - 1.\end{aligned}$$

Thus $a_2 = 2$ and $\xi_3 = (\sqrt{3}-1)^{-1}$. As $\xi_3 = \xi_1$ it follows that the continued fraction expansion is periodic:

$$\sqrt{3} = [1; 1, 2, 1, 2, \ldots].$$

The first few convergents are:

$$\frac{1}{1} \qquad \frac{2}{1} \qquad \frac{5}{3} \qquad \text{and} \qquad \frac{7}{4}.$$

9.2.3. We have

$$\begin{aligned}\xi &= \frac{\sqrt{5}+1}{2}\\ &= 1 + \frac{\sqrt{5}-1}{2}\\ &= 1 + \frac{1}{\frac{2}{\sqrt{5}-1}}\\ &= 1 + \frac{1}{\frac{\sqrt{5}+1}{2}}\\ &= 1 + \frac{1}{1+\frac{\sqrt{5}-1}{2}}\\ &= 1 + \frac{1}{1+\xi}.\end{aligned}$$

Thus
$$\xi = [1; 1, 1, 1, \dots].$$
9.2.4. $x = 2/5 + \epsilon$ and $\mathcal{F}_3$. 1/3 is closer than 1/2 to x but
$$|4/5 - 1| = 1/5 \qquad \text{and} \qquad |6/5 - 1| = 1/5.$$
so that x is not a best approximation.
9.2.7. We have
$$x = [a_0; a_1, a_2, \dots, a_k].$$
Since a/b is a best approximation to x and there is no other best approximation with larger denominator, then we must have
$$\frac{a}{b} = \frac{p_k}{q_k},$$
so that $a = p_k$ and $b = q_k$.
Since
$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k$$
it follows that $((-1)^{k+1} c q_{k-1}, (-1)^k c p_{k-1})$ is a solution of
$$ax + by = c.$$
We have to find the continued fraction expansion of 247/77. We have
$$\begin{aligned} x &= \frac{247}{77} \\ &= 3 + \frac{16}{77}. \end{aligned}$$
Thus $a_0 = 3$ and
$$\begin{aligned} x_1 &= \frac{77}{16} \\ &= 4 + \frac{13}{16}. \end{aligned}$$
Thus $a_1 = 4$ and
$$\begin{aligned} \xi_2 &= \frac{16}{13} \\ &= 1 + \frac{3}{13}. \end{aligned}$$
Thus $a_2 = 1$ and
$$\begin{aligned} \xi_3 &= \frac{13}{3} \\ &= 4 + \frac{1}{3}. \end{aligned}$$

Thus $a_3 = 3$ and $a_4 = 3$. It follows that

$$\frac{247}{77} = [3; 4, 1, 4, 3].$$

The convergents are:

$$\frac{3}{1} \qquad \frac{13}{4} \qquad \frac{16}{5} \qquad \frac{77}{24} \qquad \text{and} \qquad \frac{249}{77}.$$

This means $p_3 = 77$ and $q_3 = 24$. It follows that a solution of

$$247x + 77y = 31,$$

is

$$x = -24 \cdot 31 \qquad \text{and} \qquad y = 77 \cdot 31.$$

The general solution is then

$$x = -24 \cdot 31 + 77\lambda \qquad \text{and} \qquad y = 77 \cdot 31 - 247\lambda.$$

9.2.8. Let

$$y_k = [a_k; a_{k-1}, a_{k-2}, \dots, a_2, a_1].$$

We will show by induction on k that

$$y_k = \frac{q_k}{q_{k-1}}.$$

$p_0 = a_0$ and $q_0 = 1$.

$$q_1 = a_1 \cdot 1 + 0 = a_1.$$

We have

$$q_k = a_k q_{k-1} + q_{k-2}.$$

Thus

$$\begin{aligned} \frac{q_k}{q_{k-1}} &= a_k + \frac{q_{k-2}}{q_{k-1}} \\ &= a_k + 1/y_{k-1} \\ &= [a_k; a_{k-1}, a_{k-2}, \dots, a_2, a_1] \\ &= y_k. \end{aligned}$$

9.2.9. Let

$$\frac{a}{b} = [a_0; a_1, a_2, \dots]$$

We will show by induction on k that $a_k = q_{k+1}$ and $\xi_i = r_{i-1}/r_i$. Note that

$$a/b = q_1 + r_1/b,$$

so that $q_1 = \llcorner a/b \lrcorner = a_0$ and $b/r_1 = \xi_1$. Note that

$$r_{k-1}/r_k = q_{k+1} + r_{k+1}/r_k.$$

By induction

$$\xi_k = r_{k-1}/r_k.$$

Thus

$$q_{k+1} = \llcorner \xi_k \lrcorner = a_k.$$

It follows that

$$\xi_{k+1} = r_k / r_{k+1}.$$

This completes the induction.