MODEL ANSWERS TO THE SIXTH HOMEWORK

8.2.7. Suppose that α is a solution of $x^2 - dy^2 = 4$. We may suppose that the coefficients of α are positive. Pick a natural number n such that

$$2\left(\frac{\zeta}{2}\right)^n \leq \alpha < 2\left(\frac{\zeta}{2}\right)^{n+1}.$$

Let

$$\beta = \frac{\alpha}{\left(\frac{\zeta}{2}\right)^n}.$$

Then $2 \leq \beta < \zeta$ and $N(\beta) = 4$. As

$$4 = \zeta\bar{\zeta}$$

it follows that

$$\left(\frac{\zeta}{2}\right)^{-1} = \frac{\bar{\zeta}}{2},$$

so that

$$\beta = \alpha \cdot \left(\frac{\bar{\zeta}}{2}\right)^n \in \mathbb{Z}[\sqrt{d}].$$

Minimality of ζ implies that $\beta = 2$ so that

$$\alpha = 2\left(\frac{\zeta}{2}\right)^n.$$

Now suppose $x^2 - dy^2 = -4$ is solvable. Let η be a minimal solution with positive coefficients. Let

$$\beta = 2\left(\frac{\eta}{2}\right)^2.$$

Arguing as in the proof of Theorem 8.8, it is easy to see that $\beta \in \mathbb{Z}[\sqrt{d}]$. On the other hand,

$$\begin{aligned} N(\beta) &= N(1/2)N(\eta)N(\eta) \\ &= 4, \end{aligned}$$

so that β is a solution of $x^2 - dy^2 = 4$ with positive coefficients. By minimality of ζ It follows that

$$1 < \zeta \leq \beta.$$

As

$$-4 = \eta\bar{\eta}$$

it follows that

$$\left(\frac{\eta}{2}\right)^{-1} = -\frac{\bar{\eta}}{2}.$$

As

$$\phi = -\zeta\bar{\eta} \in \mathbb{Z}[\sqrt{d}].$$

We have

$$\eta^{-1} < \phi \leq \eta.$$

Note that

$$N(\phi) = -1,$$

and so $\phi \neq 1$. We can split the inequalities above into two inequalities

$$\eta^{-1} < \phi < 1 \qquad \text{and} \qquad 1 < \phi \leq \eta.$$

The first becomes

$$1 < \phi^{-1} < \eta$$

and so we must have $\phi = \eta$ by minimality of η.
Thus

$$\zeta = 2\left(\frac{\eta}{2}\right)^2.$$

It is easy to see that

$$2\left(\frac{\zeta}{2}\right)^n$$

is a solution of $x^2 - dy^2 = -4$. Finally suppose that α is a solution of $x^2 - dy^2 = -4$. $\eta \leq \alpha$ by minimality of η. Pick a natural number n such that

$$2\left(\frac{\zeta}{2}\right)^n \leq \alpha < 2\left(\frac{\zeta}{2}\right)^{n+1}.$$

Let

$$\beta = \alpha \cdot \left(\frac{\bar{\zeta}}{2}\right)^n \in \mathbb{Z}[\sqrt{d}].$$

Then $N(\beta) = -4$ and so

$$\eta \leq \beta < \zeta = 2\left(\frac{\eta}{2}\right)^2$$

by minimality of η. Arguing as above, it follows that $\beta = \eta$ and so

$$\alpha = \eta\left(\frac{\zeta}{2}\right)^n$$

8.2.9. Suppose $t > 0$. As

$$0 < \frac{t^2}{4}$$

it follows that

$$\begin{aligned} 1+t &< \frac{t^2}{4}+t+1 \\ &= \left(\frac{t}{2}+1\right)^2. \end{aligned}$$

Taking the square root of both sides and subtracting gives

$$\sqrt{1+t}-1 < \frac{t}{2}.$$

If we follow the same lines of the proof we get to the situation

$$u = u_1\left[x_1 - y_1\sqrt{d} + y_1\sqrt{d}\left(1-\sqrt{1+\frac{|k|}{u_1^2}}\right)\right].$$

Our goal to find a solution u whose absolute value is smaller than u_1; if u is negative, we can always flip the sign of u. Applying the result above with $t = |k|/u_1^2 > 0$ we get

$$u_1\left(\delta^{-1} - \frac{x_1|k|}{2u_1^2}\right) < u < u_1.$$

The coefficient of u_1 will be greater than -1, provided

$$\frac{x_1|k|}{2u_1^2} < 1+\delta^{-1}.$$

This is easy seen to be equivalent to

$$u_1 > \sqrt{\frac{\delta x_1|k|}{2(\delta+1)}}.$$

8.2.11. We consider the possible values of x.
If we take $x = 1$ then either $y = 3$ or $y = 4$ works:

$$|\pi - 3| = 0.14\cdots < 1 \qquad \text{and} \qquad |\pi - 4| = 0.858\cdots < 1.$$

If we take $x = 2$ then $y = 6$ works:

$$|2\pi - 6| = 0.283\cdots < 1/2.$$

If take $x = 3$ then there is no value of y, since $3\pi = 9.424\ldots$ is bigger than $9.333\ldots$ but smaller than $9.666\ldots$.
If we take $x = 4$ then there is no value of y, since $4\pi = 12.566\ldots$ is bigger than 12.25 but smaller than 12.75.
If we take $x = 5$ then there is no value of y, since $5\pi = 15.707\ldots$ is bigger than 15.2 but smaller than 15.8.
If we take $x = 6$ then $y = 19$ works:

$$|6\pi - 19| = 0.150\cdots < 1/6.$$

If we take $x = 7$ then $y = 22$ works:

$$|7\pi - 22| = 0.008 < 1/7.$$

If we take $x = 8$ then there is no value of y, since $8\pi = 25.132\dots$ is bigger than 25.125 but smaller than 25.875.
If we take $x = 9$ then there is no value of y, since $8\pi = 28.274\dots$ is bigger than $28.111\dots$ but smaller than $29.888\dots$.
If we take $x = 10$ then there is no value of y, since $10\pi = 31.415\dots$ is bigger than 31.1 but smaller than 31.9.
8.2.15. Note that

$$\begin{aligned} 2x_n &= (x_n + y_n\sqrt{d}) + (x_n - y_n\sqrt{d}) \\ &= \delta^n + \bar{\delta}^n \\ &= \delta^n + \delta^{-n}. \end{aligned}$$

Therefore

$$\begin{aligned} 4x_1x_n - 2x_{n-1} &= (\delta + \delta^{-1})(\delta^n + \delta^{-n}) - (\delta^{n-1} + \delta^{-n+1}) \\ &= \delta^{n+1} + \delta^{-(n+1)} \\ &= 2x_{n+1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2y_n &= (x_n + y_n\sqrt{d}) - (x_n - y_n\sqrt{d}) \\ &= \delta^n - \bar{\delta}^n \\ &= \delta^n - \delta^{-n}. \end{aligned}$$

Therefore

$$\begin{aligned} 4x_1y_n - 2y_{n-1} &= (\delta + \delta^{-1})(\delta^n - \delta^{-n}) - (\delta^{n-1} - \delta^{-n+1}) \\ &= \delta^{n+1} - \delta^{-(n+1)} \\ &= 2y_{n+1}. \end{aligned}$$

8.2.16. Let

$$\delta = x_1 + y_1\sqrt{p}$$

be a fundamental solution, so that

$$x_1^2 - py_1^2 = 1.$$

Reducing modulo 4, we see that x_1 is odd and y_1 is even, so that $x_1 - 1$ and $x_1 + 1$ are both even. We have

$$x_1^2 - 1 = py_1^2.$$

so that

$$\frac{x_1+1}{2} \cdot \frac{x_1-1}{2} = p\left(\frac{y_1}{2}\right)^2,$$

where the three factors are integers. Let

$$u = \frac{x_1 - 1}{2} \qquad \text{and} \qquad v = \frac{y_1}{2}.$$

Then

$$u + 1 = \frac{x_1 + 1}{2}$$

so that

$$u(u+1) = pv^2.$$

Note that every prime factor of v must divide exactly one of u or $u-1$. Suppose that

$$u + 1 = a^2 \qquad \text{and} \qquad u = pb^2,$$

for integers a and b. We may assume that a and $b \geq 0$. We have

$$x_1 = 2a^2 - 1 \qquad \text{and} \qquad x_1 = 2pb^2 + 1,$$

so that $a \leq x_1$ and $b < x_1$. It follows that

$$a^2 - pb^2 = 1.$$

As δ is the fundamental solution we must have $a = 1$ and $b = 0$, so that $x_1 = 1$, a contradiction.
Thus

$$u + 1 = pa^2 \qquad \text{and} \qquad u = b^2,$$

for integers a and b. We have

$$b^2 - pa^2 = -1.$$

8.2.17. If two integers are consecutive then they are coprime. It follows that we are looking for solutions to

$$x^2 + y^2 = z^2,$$

where x, y and z are pairwise coprime. If we assume that y is even the general solution is

$$x = c(a^2 - b^2) \qquad y = 2abc \qquad \text{and} \qquad z = c(a^2 + b^2),$$

where $c = \pm 1$, and a and b are coprime and have opposite parity. Then we want to choose a and b such that

$$a^2 - b^2 = 2ab \pm 1$$

Completing the square gives

$$(a - b)^2 - 2b^2 = \pm 1.$$

Thus we are looking for solutions to Pell's equation

$$u^2 - 2v^2 = \pm 1.$$

The minimal solution to

$$u^2 - 2v^2 = -1$$

with positive coefficients is $\gamma = 1 + \sqrt{2}$ and every solution to

$$u^2 - 2v^2 = \pm 1,$$

is then a power of γ. The general solution of Pell's equation is then given by the coefficients of $\pm\gamma^n$. As $u = a - b$ and $v = b$ it follows that $a = u + v$ and $b = v$.

If we take $u = v = 1$ then $a = 2$ and $b = 1$. This gives the Pythagorean triple $(3, 4, 5)$. If we square γ we get $u = 3$ and $v = 2$. This gives $a = 5$ and $b = 2$. In this case we get the Pythagorean triple $(21, 20, 29)$. If we cube γ this gives $u = 7$ and $v = 5$. This gives $a = 12$ and $b = 5$. In this case we get the Pythagorean triple $(119, 120, 169)$.