6. Further developments

Since the work on sums of two, three and four squares, there has been a lot of activity on generalisations of these problems.

The first suite of generalisations was introduced by Waring, 1770:

Problem 6.1 (Waring Problem). *Fix k. Is every natural number the sum of a fixed number of kth powers?*

Supposing the answer is affirmative for k, let $g(k)$ denote the minimum number. For example, since every natural number is the sum of four squares but not three, the answer is yes for $k = 2$ and $g(4) = 2$.

Hilbert showed the existence of $g(k)$ for every k in 1909. Hardy and Littlewood and then Vinogradov developed different methods and showed the existence of an asymptotic formula for large natural numbers n. In other words, they considered $G(k)$, the minimal number of kth powers needed to represent a sufficiently large natural number n.

We already saw that $g(2) = G(2) = 4$. By definition $G(k) \leq g(k)$ and most of the time $G(k) < g(k)$. For example, $g(3) = 9$ and $4 \leq G(3) \leq 7$. It is interesting we don't know the correct value for $G(k)$. One interesting open problem is whether or not $G(3) = 4$, that is, whether or not every sufficiently large natural number is the sum of four cubes.

23 is the sum of 9 cubes but not 8. The largest integer known to require 6 cubes is 1290740 and the largest integer known to require 5 cubes is 7373170279850.

In a completely different direction, one can look at quadratic polynomials and see what values they take on. Let

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j,$$

be a quadratic polynomial in the variables $x_1, x_2, \ldots, x_n$. It is natural to require that $Q(x_1, x_2, \ldots, x_n) > 0$ whenever the entries are real and non-zero. We call $Q(x_1, x_2, \ldots, x_n)$ a **positive definite quadratic form** when this holds. We say that $Q(x_1, x_2, \ldots, x_n)$ is **integral** if $Q(x_1, x_2, \ldots, x_n)$ is an integer whenever $(x_1, x_2, \ldots, x_n)$ is an integer.

We call a quadratic form **universal** if it takes on all natural numbers. That is, given a natural number m, there are integers $(x_1, x_2, \ldots, x_n)$ such that $Q(x_1, x_2, \ldots, x_n) = m$.

We have the following result of Bhargava and Hanke:

Theorem 6.2 (290 Theorem). *A positive definite integral quadratic form is universal if and only if it takes on all natural numbers up to 290.*

For example, using the 290 Theorem, to show that every natural number is a sum of four squares, it suffices to show that every natural number up to 290 is a sum of four squares. One applies the 290 Theorem to

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

In fact one can get away with checking that $Q(x_1, x_2, \ldots, x_n)$ takes on the values

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290.$$

For each number in the sequence there is a quadratic form which takes on every value except that number.