20. Pell equation revisited

Theorem 20.1. *If N is an integer and d is a natural number such thar $|N| < \sqrt{d}$ and d is not a square, then for all positive solutions of the Pell equation*

$$x^2 - dy^2 = N$$

x/y is a convergent of the continued fraction expansion of $\sqrt{d}$.

Proof. If N is positive then

$$\begin{aligned} 0 &< x - y\sqrt{d} \\ &= \frac{N}{x + y\sqrt{d}} \\ &< \frac{\sqrt{d}}{x + y\sqrt{d}} \\ &= \frac{1}{(x/\sqrt{d}) + y} \\ &= \frac{1}{y(x/y\sqrt{d} + 1)}. \end{aligned}$$

Since $x/y > \sqrt{d}$, we have

$$\left|\sqrt{d} - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

It follows that x/y is a convergent of the continued expansion of $\sqrt{d}$.

Now suppose that N is negative. Using the equation

$$y^2 - \frac{x^2}{d} = -\frac{N}{d}$$

we have

$$\begin{aligned} 0 &< y - \frac{x}{\sqrt{d}} \\ &= \frac{-N/d}{y + x/\sqrt{d}} \\ &< \frac{1}{x + y\sqrt{d}} \\ &= \frac{1}{x(1 + y\sqrt{d}/x)}. \end{aligned}$$

Therefore

$$\left|\frac{1}{\sqrt{d}} - \frac{y}{x}\right| < \frac{1}{2x^2}.$$

It follows that y/x is a convergent of the continued expansion of $1/\sqrt{d}$. If

$$\sqrt{d} = [a_0; a_1, a_2, \dots] \qquad \text{then} \qquad 1/\sqrt{d} = [0; a_0, a_1, a_2, \dots].$$

Thus the convergents of $1/\sqrt{d}$ are $0/1$ and the reciprocals of the convergents of $\sqrt{d}$. □

Using continued fractions to find solutions of Pell's equation is particularly efficient as:

Proposition 20.2. *The sequence*

$$p_k^2 - dq_k^2$$

is eventually periodic, where p_k/q_k are the convergents of $\xi = \sqrt{d}$.

Proof. We have

$$\sqrt{d} = \frac{p_{k-1}\xi_k + p_{k-2}}{q_{k-1}\xi_k + q_{k-2}}.$$

Solving for ξ_k, we can write

$$\xi_k = \frac{\sqrt{d} + r_k}{s_k}$$

for rational numbers r_k and s_k. Substituting back into the previous equation and replacing the index k by the index $k+1$ we get

$$\sqrt{d} = \frac{p_k(\sqrt{d} + r_{k+1}) + p_{k-1}s_{k+1}}{q_k(\sqrt{d} + r_{k+1}) + q_{k-1}s_{k+1}},$$

so that

$$(q_k r_{k+1} + q_{k-1}s_{k+1} - p_k)\sqrt{d} - (p_{k-1}s_{k+1} + p_k r_{k+1} - q_k d) = 0.$$

Both the rational and the irrational parts must be equal to zero. Therefore

$$\begin{aligned} q_k r_{k+1} + q_{k-1}s_{k+1} &= p_k \\ p_k r_{k+1} + p_{k-1}s_{k+1} &= q_k d. \end{aligned}$$

Viewing r_{k+1} and s_{k+1} as unknowns, the determinant of the coefficient matrix is

$$q_k p_{k-1} - q_{k-1}p_k = (-1)^k.$$

It follows that

$$\begin{aligned} r_{k+1} &= (-1)^k(p_k p_{k-1} - q_k q_{k-1}d) \\ s_{k+1} &= (-1)^k(q_k^2 d - p_k^2). \end{aligned}$$

As the numbers r_k and s_k are determined by ξ_k, and ξ_k is eventually periodic, it follows that r_k and s_k are eventually periodic. □

This gives us an effective method to find the solutions of Pell's equation. The first time we find a convergent which is a solution of $x^2 - dy^2 = 1$, we must have found the fundamental solution, since this will have the smallest numerator and denominator. If $x^2 - dy^2 = -1$ has a solution, then we will see this solution before the second period (we need to go the second period, due to the factor $(-1)^k$.

For example, consider

$$x^2 - 21y^2 = N.$$

We have to write down the continued fraction expansion of $\sqrt{21}$.