19. Quadratic irrationalities

Even though in theory continued fractions completely answer the question of finding best approximations, in practice it is not so easy to spot patterns in the continued fraction expansion of a real number. There are some interesting exceptions.

It is possible to show that

$$2.7182 < e < 2.7183.$$

One can easily compute

$$2.7182 = [2; 1, 2, 1, 1, 4, 1, 1, 1, 3, 1, 9].$$

and

$$2.7183 = [2; 1, 2, 1, 1, 4, 1, 1, 19, 1, 1, 3].$$

It follows that

$$e = [2; 1, 2, 1, 1, 4, 1, 1, \dots].$$

It is in fact known that the continued fraction expansion of e has the pattern

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots, 1, 2n, 1, \dots].$$

There is one simple case where it is possible to calculate the continued fraction expansion. For example, suppose

$$\xi = [4; 3, 1, 2, 1, 2] = [4; 3, \overline{1, 2}].$$

Note that

$$\xi = [4; 3, \xi_2] \qquad \text{where} \qquad \xi_2 = \tau = [\overline{1; 2}].$$

Note that

$$\begin{aligned} \tau &= 1 + \frac{1}{2 + 1/\tau} \\ &= 1 + \frac{\tau}{2\tau + 1} \\ &= \frac{3\tau + 1}{2\tau + 1}. \end{aligned}$$

It follows that

$$2\tau^2 - 2\tau - 1 = 0.$$

It follows that

$$\tau = \frac{1 + \sqrt{3}}{2}.$$

From there we can figure out ξ. We have

$$\begin{aligned}
\xi &= 4 + \cfrac{1}{3 + \cfrac{1}{\frac{\sqrt{3}+2}{2}}} \\
&= 4 + \cfrac{1}{3 + \cfrac{2}{\sqrt{3}+1}} \\
&= 4 + \frac{1}{\frac{3\sqrt{3}+5}{\sqrt{3}+1}} \\
&= 4 + \frac{\sqrt{3}+1}{3\sqrt{3}+5} \\
&= \frac{13\sqrt{3}+21}{3\sqrt{3}+5} \\
&= \frac{(13\sqrt{3}+21)(3\sqrt{3}-5)}{2} \\
&= 6 - \sqrt{3}.
\end{aligned}$$

We have already seen examples of how square roots gives rise to eventually periodic continued fractions. We say a real α is a **quadratic irrational** if α has degree two over $\mathbb{Q}$.

Theorem 19.1. *ξ is a quadratic irrational if and only if its continued fraction expansion is eventually periodic.*

Proof. Suppose that the continued fraction expansion of ξ is eventually periodic,

$$\begin{aligned}
\xi &= [a_0; a_1, a_2, \dots] \\
&= [a_0; a_1, a_2, \dots, a_{n-1}, \overline{a_n, a_{n+1}, \dots a_{n+h-1}}] \\
&= [a_0; a_1, a_2, \dots, a_{n-1}, \xi_n],
\end{aligned}$$

where $\xi_n = \xi_{n+h}$. Then

$$\begin{aligned}
\xi &= \frac{p_{n-1}\xi_n + p_{n-2}}{q_{n-1}\xi_n + q_{n-2}} \\
&= \frac{p_{n+h-1}\xi_n + p_{n+h-2}}{q_{n+h-1}\xi_n + q_{n+h-2}}.
\end{aligned}$$

Solving for ξ_n gives a quadratic equation for ξ_n with rational coefficients,

$$A\xi_n^2 + B\xi_n + C = 0.$$

If we substitute

$$\xi_n = \frac{-q_{n-2}\xi + p_{n-2}}{q_{n-1}\xi - p_{n-1}}$$

into this equation, expand and clear denominators, this gives us a quadratic equation with rational coefficients for ξ. Thus ξ has degree at most two. As ξ is not rational it has degree two, that is, ξ is a quadratic irrational.

Now suppose that ξ is a quadratic irrational so that ξ is irrational and satisfies a quadratic equation

$$A\xi^2 + B\xi + C = 0,$$

with integer coefficients. Subsituting and clearing denominators gives:

$$A(p_{k-1}\xi_k+p_{k-2})^2+B(p_{k-1}\xi_k+p_{k-2})(q_{k-1}\xi_k+q_{k-2})+C(q_{k-1}\xi_k+q_{k-2})^2 = 0.$$

Multiplying out gives

$$A_k\xi_k^2 + B_k\xi_k + C_k = 0,$$

where

$$\begin{aligned} A_k &= Ap_{k-1}^2 + Bp_{k-1}q_{k-1} + Cq_{k-1}^2 \\ B_k &= 2Ap_{k-1}p_{k-2} + B(p_{k-1}q_{k-2} + p_{k-2}q_{k-1}) + Cq_{k-1}q_{k-2} \\ C_k &= Ap_{k-2}^2 + Bp_{k-2}q_{k-2} + Cq_{k-2}^2. \end{aligned}$$

If $f(x) = Ax^2 + Bx + C$ then

$$A_k = q_{k-1}^2 f\left(\frac{p_{k-1}}{q_{k-1}}\right) \qquad \text{and} \qquad C_k = q_{k-2}^2 f\left(\frac{p_{k-2}}{q_{k-2}}\right)$$

By Taylor's theorem

$$A_k = q_{k-1}^2 \left[f(\xi) + f'(\xi)\left(\frac{p_{k-1}}{q_{k-1}} - \xi\right) + \frac{1}{2}f''(\xi)\left(\frac{p_{k-1}}{q_{k-1}} - \xi\right)^2 \right].$$

As

$$\left|\xi - \frac{p_{k-1}}{q_{k-1}}\right| < \frac{1}{q_{k-1}^2},$$

it follows that A_k is bounded. Similarly C_k is bounded. Now we check B_k is bounded.

We use the fact that the discriminant

$$B_k^2 - 4A_kC_k = D$$

is independent of k. One can either check this directly or use the fact that the quadratic form

$$A_ku^2 + B_kuv + C_kv^2$$

is obtained from the quadratic form

$$Ax^2 + Bxy + Cy^2$$

by the linear transformation

$$\begin{aligned} x &= p_{k-1}u + p_{k-2}v \\ y &= q_{k-1}u + q_{k-2}v. \end{aligned}$$

As the determinant of this linear transformation is ± 1 this implies that the discriminant is unchanged.

Since

$$B_k^2 = D + 4A_kC_k,$$

where D is unchanged and A_k and C_k are bounded, it follows that B_k is bounded.

As all three of A_k and B_k and C_k are bounded, it follows that there are only finitely many different triples. As there are infinitely many possible choices of k, the pigeonhole principle implies that there are three indices n_1, n_2 and n_3 such that ξ_{n_1}, ξ_{n_2}, and ξ_{n_3} all satisfy the same quadratic equation. Since one quadratic equation has at most two roots, possibly relabelling, we must have $\xi_{n_1} = \xi_{n_2}$, where $n_1 < n_2$. It follows that the continued fraction expansion of ξ is periodic starting at $k = n_1$ with period $h = n_2 - n_1$. □

Definition 19.2. *We say that the quadratic irrational ξ is **reduced** if $\xi > 1$ and its conjugate $-1 < \bar{\xi} < 0$.*

Theorem 19.3. *A quadratic irrational ξ is reduced if and only if*

$$\xi = [\overline{a_0; a_1, a_2, \dots, a_{h-1}}].$$

Proof. Suppose that

$$\begin{aligned} \xi &= [a_0; a_1, a_2, \dots, a_{l-1}, \overline{a_l, a_{l+1}, \dots, a_{l+h}}] \\ &= [a_0; a_1, a_2, \dots, a_{l-1}, \xi_l], \end{aligned}$$

where $\xi_l = \xi_{l+h}$. We assume that l is the smallest index with this property. Our goal is to characterise when $l = 0$.

By assumption

$$\xi_k = a_k + \frac{1}{\xi_{k+1}} \qquad \text{and} \qquad a_k = \llcorner \xi_k \lrcorner.$$

Taking conjugates, we get

$$-\frac{1}{\bar{\xi}_{k+1}} = a_k - \bar{\xi}_k.$$

Let

$$\eta_k = (-\bar{\xi}_k)^{-1}.$$

Then ξ is reduced if and only if $\xi_0 > 1$ and $\eta_0 > 1$. Note that

$$\eta_{k+1} = a_k + \frac{1}{\eta_k}.$$

Suppose that ξ is reduced. As $\eta_0 > 1$ it follows that $\eta_k > 1$ so that

$$a_k = \llcorner \eta_{k+1} \lrcorner.$$

Suppose that $l > 0$. Then $\eta_{l+h} = \eta_l$ and so $a_{l+h-1} = a_{l-1}$. It follows that $\eta_{l+h-1} = \eta_{l-1}$ so that $\xi_{l+h-1} = \xi_{l-1}$, which contradicts minimality of l. Therefore, $l = 0$ if ξ is reduced.

Now suppose that $l = 0$. It follows that $a_0 > 0$ as $a_h > 0$. Thus $\xi > 1$ and $\xi = \xi_h$. We have

$$\begin{aligned}
\eta &= \eta_h \\
&= [a_{h-1}; \eta_{h-1}] \\
&\vdots \\
&= [a_{h-1}; a_{h-2}, \dots, a_1, a_0, \eta] \\
&= [\overline{a_{h-1}; a_{h-2}, \dots, a_1, a_0}] \\
&> 1,
\end{aligned}$$

and so ξ is reduced. $\square$

Corollary 19.4. *Suppose that $r > 1$ is a rational number, not the square of a rational number.*

Then

$$\sqrt{r} = [a_0; \overline{a_1, a_2, \dots, a_{h-1}, 2a_0}],$$

where $a_1, a_2, \dots, a_{h-1}$ has central symmetry: $a_1 = a_{h-1}$, $a_2 = a_{h-2}$, ...

Proof. Let $\xi = \sqrt{r} + \llcorner \sqrt{r} \lrcorner$. Then $\xi > 1$ and

$$-1 < \bar{\xi} = \llcorner \sqrt{r} \lrcorner - \sqrt{r} < 0,$$

so that ξ is reduced.

On the other hand,

$$\begin{aligned}
\eta &= (-\bar{\xi})^{-1} \\
&= (\sqrt{r} - \llcorner \sqrt{r} \lrcorner)^{-1} \\
&= \xi_1.
\end{aligned}$$

This implies that if

$$\xi = [\overline{b_0; b_1, b_2, \dots, b_h}]$$

then the sequence $b_0, b_1, b_2, \ldots, b_h$ has central symmetry. On the other hand

$$\begin{aligned} \llcorner \xi \lrcorner &= \llcorner \sqrt{r} \lrcorner + \llcorner \sqrt{r} \lrcorner \\ &= 2 \llcorner \sqrt{r} \lrcorner, \end{aligned}$$

so that $b_0 = 2a_0$ and otherwise $b_i = a_i$. $\square$