## 18. Infinite Continued fractions

Suppose $x = \xi$ is an irrational number. Then we have an infinite continued fraction expansion for $\xi$,

$$\xi = [\lambda_0; \lambda_1, \lambda_2, \dots].$$

Note that

$$\frac{p_k}{q_k} = [\lambda_0; \lambda_1, \lambda_2, \dots, \lambda_k]$$

are best approximations of $\xi$ so that

$$\begin{aligned} \xi &= \lim_{k\to\infty} \frac{p_k}{q_k} \\ &= \lim_{k\to\infty} [\lambda_0; \lambda_1, \lambda_2, \dots, \lambda_k]. \end{aligned}$$

Now we check we can reverse all of this.

Suppose we start with an arbitrary continued fraction

$$[a_0; a_1, a_2, \dots].$$

Consider the congergents:

$$\frac{p_0}{q_0} = a_0, \qquad \frac{p_1}{q_1} = [a_0; a_1], \qquad \frac{p_2}{q_2} = [a_0; a_1, a_2], \dots.$$

Pick $n > 2$. Then the numbers above for $k \leq n-1$ are the convergents of

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n].$$

It follows that

$$\frac{p_{2k-2}}{q_{2k-2}} < \frac{p_{2k}}{q_{2k}} \qquad \text{and} \qquad \frac{p_{2k-1}}{q_{2k-1}} > \frac{p_{2k+1}}{q_{2k+1}},$$

so that

$$\frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \dots, \qquad \text{and} \qquad \frac{p_1}{q_1} > \frac{p_3}{q_3} > \frac{p_5}{q_5} > \dots.$$

On the other hand,

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2k+1}}{q_{2k+1}}.$$

In particular the sequence of even convergents is monotonic increasing and bounded above by any of the odd convergents; similarly the odd convergents are monotonic decreasing and bounded below by any of the even convergents. It follows that the even and odd convergents both tend to a limit; we check they tend to the same limit.

We have

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k-1}q_k}.$$

As the $q_1, q_2, \dots$ is strictly increasing the RHS tends to zero and the convergents tend to the same limit $\xi$.

Let's try all of this out with $\xi = \sqrt{11}$. $\sqrt{11} = 3 + (\sqrt{11} - 3)$, so that $a_0 = 3$ and $\xi_1 = (\sqrt{11} - 3)^{-1}$.

$$\begin{aligned} \frac{1}{\sqrt{11} - 3} &= \frac{\sqrt{11} + 3}{2} \\ &= 3 + \frac{\sqrt{11} - 3}{2}, \end{aligned}$$

so that $a_1 = 3$ and

$$\xi_2 = \left( \frac{\sqrt{11} - 3}{2} \right)^{-1}.$$

We have

$$\begin{aligned} \frac{2}{\sqrt{11} - 3} &= \sqrt{11} + 3 \\ &= 6 + \sqrt{11} - 3. \end{aligned}$$

so that $a_2 = 6$ and

$$\xi_3 = \left( \sqrt{11} - 3 \right)^{-1}.$$

As this is the same as $\xi_1$, it follows that the continued fraction is periodic starting with $a_1$ and it follows that

$$\sqrt{11} = [3; 3, 6, 3, 6, 3, 6, \dots].$$

The convergents are

$$\frac{3}{1}, \qquad \frac{10}{3}, \qquad \frac{63}{19}, \qquad \dots.$$

As expected, the convergents provide excellent approximations:

**Theorem 18.1.** *If $\xi$ is irrational then*

$$\xi - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k(q_k \xi_{k+1} + q_{k-1})}.$$

*Hence*

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| \xi - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}.$$

*In particular*

$$\left| \xi - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}.$$

*Proof.* The first equality follows from the definitions. The second inequalities follow from

$$\begin{aligned} q_{k+1} &= q_k a_{k+1} + q_{k-1} \\ &< q_k \xi_{k+1} + q_{k-1} \\ &< q_k(a_{k+1}+1) + q_{k-1} \\ &= q_k + q_{k+1}. \end{aligned}$$

□

It is interesting to observe that there is a partial converse of this result:

**Theorem 18.2.** *If $x \in \mathbb{R}$ and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

*then $p/q$ is a convergent of the continued fraction expansion of $x$.*

*Proof.* Suppose that

$$0 < x - \frac{p}{q} < \frac{1}{2q^2}.$$

The case

$$0 < \frac{p}{q} - x < \frac{1}{2q^2}$$

is handled in a similar fashion.

We show that $p/q$ is a best approximation to $x$. Suppose not. Then we may find $r/s$ such that

$$|sx - r| < |qx - p| \qquad \text{where} \qquad s \leq q.$$

Clearly

$$\left| \frac{p}{q} - \frac{r}{s} \right| \geq \frac{1}{qs}.$$

There are three cases. First suppose that

$$\frac{r}{s} < \frac{p}{q} < x.$$

In this case

$$\begin{aligned} 0 &< \frac{p}{q} - \frac{r}{s} \\ &< x - \frac{r}{s} \\ &< \frac{q}{s}\frac{1}{2q^2} \\ &= \frac{1}{2qs}, \end{aligned}$$

a contradiction.

Now suppose that

$$\frac{p}{q} < \frac{r}{s} < x.$$

In this case

$$\begin{aligned} 0 &< \frac{r}{s} - \frac{p}{q} \\ &< x - \frac{p}{q} \\ &< \frac{1}{2q^2} \\ &< \frac{1}{qs}, \end{aligned}$$

a contradiction.

Finally suppose that

$$\frac{p}{q} < x < \frac{r}{s}.$$

Observe that

$$\begin{aligned} \frac{r}{s} - x &= \frac{r - sx}{s} \\ &< \frac{qx - p}{s} \\ &= \left(x - \frac{p}{q}\right) \cdot \frac{q}{s}. \end{aligned}$$

Hence

$$\begin{aligned} 0 &< \frac{r}{s} - \frac{p}{q} \\ &= \frac{r}{s} - x + x - \frac{p}{q} \\ &< \left(1 + \frac{q}{s}\right)\left(x - \frac{p}{q}\right) \\ &< \frac{q+s}{s \cdot 2q^2} \\ &\leq \frac{1}{qs}, \end{aligned}$$

a contradiction. □