It is expedient to add two more terms. Let $p_{-2} = 0$ and $q_{-2} = 1$ and $p_{-1} = 1$, $q_{-1} = 0$. Note that the formula

$$q_{k+1}p_k - q_k p_{k+1} = (-1)^{k+1}.$$

in (16.3) still continues to hold, even though these initial numbers don't have much to do with finding a good approximation of x.

Consider the linear Diophantine equation

$$p_k v - q_k u = (-1)^{k+1}.$$

By (16.3) $v = q_{k+1}$ and $u = p_{k+1}$ is one solution. On the other hand, $v = q_{k-1}$ and $u = p_{k-1}$ is another solution. It follows that there is an integer λ_{k+1} such that

$$\begin{aligned} p_{k+1} &= p_{k-1} + \lambda_{k+1} p_k \\ q_{k+1} &= q_{k-1} + \lambda_{k+1} q_k. \end{aligned}$$

Observe that λ_{k+1} is a natural number as $q_{k+1} > q_k$. Multiplying the second equation by x and subtracting the first equation gives

$$\alpha_{k+1} = \alpha_{k-1} + \lambda_{k+1}\alpha_k.$$

Since the sign is alternating, this gives

$$|\alpha_{k-1}| = |\alpha_{k+1}| + \lambda_{k+1}|\alpha_k|.$$

Dividing through by $|\alpha_k|$ gives

$$\left|\frac{\alpha_{k-1}}{\alpha_k}\right| = \lambda_{k+1} + \left|\frac{\alpha_{k+1}}{\alpha_k}\right|.$$

Since

$$-1 \leq \frac{\alpha_{k+1}}{\alpha_k} < 0,$$

it follows that

$$\lambda_{k+1} = \llcorner -\frac{\alpha_{k-1}}{\alpha_k} \lrcorner.$$

This gives us a simple way to compute p_k and q_k in terms of $\alpha_1, \alpha_2, \dots$ and $\lambda_1, \lambda_2, \dots$.

In practice it is best to express everything in terms of

$$x_{k+1} = -\frac{\alpha_{k-1}}{\alpha_k}.$$

In fact

$$\alpha_{-1} = 0 \cdot x - 1 = -1 \qquad \text{and} \qquad \alpha_0 = 1 \cdot x - \llcorner x \lrcorner = \{\, x \,\},$$

so that

$$x_1 = \frac{1}{x - \llcorner x \lrcorner}.$$

More generally,

$$\begin{aligned} x_{k+1} &= -\frac{\alpha_{k-1}}{\alpha_k} \\ &= -\frac{\alpha_{k-1}}{\lambda_k \alpha_{k-1} + \alpha_{k-2}} \\ &= \frac{1}{-\alpha_{k-2}/\alpha_{k-1} - \lambda_k}. \end{aligned}$$

Therefore

$$x_k = \frac{1}{x_k - \llcorner x_k \lrcorner}.$$

Let's see how this works for $x = \pi$.

$$\lambda_0 = \llcorner \pi \lrcorner = 3.$$

and so

$$\begin{aligned} x_1 &= \frac{1}{x - \llcorner x \lrcorner} \approx 7.06, & \lambda_1 &= \llcorner x_1 \lrcorner = 7 \\ x_2 &= \frac{1}{x_1 - \llcorner x_1 \lrcorner} \approx 15.99, & \lambda_2 &= \llcorner x_2 \lrcorner = 15 \\ x_3 &= \frac{1}{x_2 - \llcorner x_2 \lrcorner} \approx 1.00, & \lambda_3 &= \llcorner x_3 \lrcorner = 1 \\ x_4 &= \frac{1}{x_3 - \llcorner x_3 \lrcorner} \approx 292.62, & \lambda_4 &= \llcorner x_4 \lrcorner = 292. \end{aligned}$$

The best approximations of π are then

$$\frac{3}{1} \qquad \frac{22}{7} \qquad \frac{333}{106} \qquad \frac{355}{113} \qquad \text{and} \qquad \frac{103,993}{33,102}$$

It is convenient to present this data in a slightly different way. We have

$$x_1 = \frac{1}{x - \lambda_0} \qquad x_2 = \frac{1}{x_1 - \lambda_1} \qquad \ldots.$$

Solving for x gives and then for x_1 and so on, gives

$$\begin{aligned} x &= \lambda_0 + \frac{1}{x_1} \\ &= \lambda_0 + \cfrac{1}{\lambda_1 + \cfrac{1}{x_2}} \\ \vdots &= \vdots \\ &= \lambda_0 + \cfrac{1}{\lambda_1 + \cfrac{1}{\lambda_2 + \dots + \cfrac{1}{\lambda_{k-1}} + \cfrac{1}{x_k}}} \end{aligned}$$

There is a very similar expression for p_k/q_k except that the last term is λ_k and not x_k:

$$\frac{p_k}{q_k} = \lambda_0 + \cfrac{1}{\lambda_1 + \cfrac{1}{\lambda_2 + \dots + \cfrac{1}{\lambda_{k-1}} + \cfrac{1}{\lambda_k}}}.$$

It is convenient to represent a continued fraction using much more compact notation:

$$x = [\lambda_0; \lambda_1, \lambda_2, \dots, \lambda_{k-1}, x_k] \qquad \text{and} \qquad \frac{p_k}{q_k} = [\lambda_0; \lambda_1, \lambda_2, \dots, \lambda_k].$$

Let κ be the smallest index such that $x_k = \lambda_k$. Of course this can only happen if x is rational. If x is irrational, so that $x_k \neq \lambda_k$, then we put $\kappa = \infty$.

Up to now we have ignored the annoying possibility that for some k we can find r and s with $s > q_{k-1}$, $(r, s) = 1$ and

$$|sx - r| = |q_{k-1}x - p_{k-1}|.$$

Since $r/s \neq p_{k-1}/q_{k-1}$ we must have

$$sx - r = -(q_{k-1}x - p_{k-1}).$$

It follows that

$$x = \frac{p_{k-1} + r}{q_{k-1} + s},$$

so that x is the mediant between p_{k-1}/q_{k-1} and r/s in the Farey sequence $\mathcal{F}_s$. It follows that $k = \kappa$.

We can choose to include r/s as a best approximation or not. If we include it then we take $r/s = p_k/q_k$ and $p_{k+1}/q_{k+1} = x$. In this case $\lambda_{\kappa+1} = 1$ and so

$$x = [\lambda_0; \lambda_1, \lambda_2, \ldots, \lambda_k, 1].$$

If we choose not to include r/s then $x = p_\kappa/q_\kappa$ and

$$x = [\lambda_0; \lambda_1, \lambda_2, \ldots, \lambda_k + 1].$$

Example 17.1.

$$\frac{4}{3} = [1; 2, 1] = [1; 3].$$

Finally, note that if we are given an arbitrary finite continued fraction

$$[a_0; a_1, a_2, \ldots, a_n],$$

it has a rational value and the fractions

$$\frac{p_0}{q_0} = a_0, \qquad \frac{p_1}{q_1} = [a_0; a_1], \qquad \frac{p_2}{q_2} = [a_0; a_1, a_2], \ldots,$$

are called the **convergents** of the continued fraction. Note that

$$x = [a_0; a_1, a_2, \ldots, a_{k-1}, x_k] \qquad \text{where} \qquad x_k = [a_k; a_{k+1}, \ldots, a_n],$$

so that,

$$x_k = a_k + \frac{1}{x_{k+1}}.$$

It is then easy to see that $x_1, x_2, \ldots, x_n$ is the sequence we constructed before.

Note that x has a unique continued fraction expression, up to replacing the last term a_n with the two terms $a_n - 1$ and 1.