16. Best approximations

$$\frac{22}{7}$$

is a very common approximation of π as it is both simple and it is also a remarkably good approximation; the error is about 0.00126. The next best approximation is

$$\frac{179}{57},$$

when the error is 0.00124. Two other very good approximations (known in Europe since the 16th century) are

$$\frac{333}{106}$$

where the error is about 8.3×10^{-5} and

$$\frac{335}{113}$$

where the error is about 2.6×10^{-7} (known in China since the 5th century).

No fraction with denominator $106 < q < 113$ is better than 22/7. It is the aim of this section to examine this sort of phenomena.

Definition 16.1. *We say that p/q is a **best approximation** to x if*

$$|q'x - p'| \leq |qx - p|$$

implies that $q' \geq q$ and if $q' = q$ then $p' = p$.

Lemma 16.2. *If p/q is a best approximation of x then p/q is the closest element of $\mathcal{F}_q$ to x.*

Proof. If $p'/q' \in \mathcal{F}_q$ then $q' \leq q$ and so if $p/q \neq p'/q'$ we have

$$\begin{aligned} \left| x - \frac{p}{q} \right| &< \frac{q'}{q} \left| x - \frac{p'}{q'} \right| \\ &\leq \left| x - \frac{p'}{q'} \right|. \end{aligned}$$

□

Note that the converse of (16.2) need not hold; the closest element p/q of $\mathcal{F}_q$ need not be a best approximation.

Observe that if $p = mp'$ and $q = mq'$ and $m > 1$ then

$$\begin{aligned} |q'x - p'| &= m|qx - p| \\ &< |qx - p|. \end{aligned}$$

Thus if p and q gives a best approximation then p and q are automatically coprime. Note that this gives us an algorithm to construct a sequence of rational numbers p_k/q_k. Just keep taking the next best approximation.

First note that the Farey sequence $\mathcal{F}_n$ is periodic with period one; if you know the rational numbers belonging to $[0,1)$ just shift these by the integers to get the whole Farey sequence.

So the first step is to shift x into the interval $[0,1)$. Put $p_0 = \llcorner x \lrcorner$ and $q_0 = 1$. Note that p_0/q_0 need not be a best approximation; in fact it is a best approximation if and only if $\{ x \} < 1/2$.

Note that given a positive real $\tau > 0$ there are integers p and $q \in \mathbb{Z}$ such that

$$1 \leq q \leq \tau \qquad \text{and} \qquad |qx - p| < \frac{1}{\tau}.$$

If $q_0 x - p_0 = 0$ then $x = p_0/q_0$ and we stop the algorithm. If not, put

$$\tau = |q_0 x - p_0|^{-1}$$

and then pick the smallest q

$$1 \leq q \leq \tau \qquad \text{such that} \qquad |qx - p| < |q_0 x - p_0|.$$

This defines the pair p_1 and q_1 (if p_0/q_0 is not a best approximation then $q_1 = q_0 = 1$, but this is the only time this will happen). If $x = p_1/q_1$ then stop. Otherwise let

$$q_1 < q_2 \leq |q_1 x - p_1|^{-1}$$

be the smallest integer such that

$$|q_2 x - p_2| < |q_1 x - p_1|$$

for some integer p_2. Continuing in this fashion, for those integers k such that $q_k x - p_k \neq 0$, we construct integers

$$q_0 \leq q_1 < q_2 < \dots,$$

such that

$$|q_{k+1} x - p_{k+1}| < |q_k x - p_k|$$

and

$$|qx - p| \geq |q_k x - p_k| \qquad \text{for all} \qquad 0 < q < q_{k+1}, \quad \forall p.$$

Further

$$q_{k+1} \leq |q_k x - p_k|^{-1}.$$

In principle this gives an algorithm. However it would be nice to give a better description of how to find q_k. The key result is:

Lemma 16.3.

$$q_{k+1} p_k - q_k p_{k+1} = (-1)^{k+1}.$$

Proof. We start by proving the weaker assertion that the RHS is ± 1. Note that

$$q_{k+1}p_k - q_k p_{k+1} = q_k(q_{k+1}x - p_{k+1}) - q_{k+1}(q_k x - p_k).$$

Set

$$\alpha_k = q_k x - p_k.$$

Then

$$\begin{aligned} |q_{k+1}p_k - q_k p_{k+1}| &\le q_k|\alpha_{k+1}| + q_{k+1}|\alpha_k| \\ &< 2q_{k+1}|\alpha_k| \\ &< 2. \end{aligned}$$

As $q_{k+1}p_k - q_k p_{k+1}$ is an integer, it follows that

$$q_{k+1}p_k - q_k p_{k+1} = \pm 1.$$

Suppose that α_k and α_{k+1} have the same sign. Then

$$\begin{aligned} |\alpha_k| &> |\alpha_k - \alpha_{k+1}| \\ &= |(q_{k+1} - q_k)x - (p_{k+1} - p_k)|. \end{aligned}$$

As $q_{k+1} - q_k < q_{k+1}$ this contradicts our choice of q_{k+1}. Thus α_k alternate sign. Using the relation,

$$q_{k+1}p_k - q_k p_{k+1} = q_k\alpha_k - q_{k+1}\alpha_{k+1},$$

it follows that

$$q_{k+1}p_k - q_k p_{k+1} = \operatorname{sgn}\alpha_{k+1}$$ [1].

As $\alpha_0 = 1 \cdot x - \llcorner x \lrcorner > 0$ it follows that $\operatorname{sgn}\alpha_0 = 1$, so that $\operatorname{sgn}\alpha_k = (-1)^k$. ∎

[1] where $\operatorname{sgn} y$ is the sign, -1, 0 or 1, of y.