15. Farey Sequences

Definition 15.1. *Fix a natural number n.*

The ***Farey sequence*** *of order n, denoted $\mathcal{F}_n$, is the set of rational numbers p/q with denominator $1 \leq q \leq n$, arranged in increasing order.*

$\mathcal{F}_1$ is the sequence of integers, and so on.

Lemma 15.2. *If n is a natural number then*

$$|\mathcal{F}_n \cap [0,1]| = 1 + \varphi(1) + \varphi(2) + \cdots + \varphi(n).$$

Proof. Indeed an element of

$$\mathcal{F}_n \cap (0,1)$$

has the unique form a/b, where $2 \leq b \leq n$ and a is coprime to b. □

Definition 15.3. *p/q and $r/s \in \mathcal{F}_n$ are* ***adjacent*** *if they are successive elements of the sequence $\mathcal{F}_n$.*

Definition-Proposition 15.4.

(1) *If p/q and r/s are adjacent in $\mathcal{F}_n$ for some n then $|ps - qr| = 1$.*
(2) *If $|ps - qr| = 1$ then p/q and r/s are adjacent in $\mathcal{F}_n$ for*

$$\max(q, s) \leq n < q + s.$$

and they are separated by the single element

$$\frac{(p+r)}{(q+s)} \in \mathcal{F}_{q+s},$$

called the ***mediant*** *of p/q and r/s.*

Proof. We first prove (2). Suppose that p/q and r/s are two elements of $\mathcal{F}_n$ such that $qr - ps = \pm 1$. Possibly switching p/q and r/s we may assume that $r/s > p/q$ and $qr - ps = 1$.

Consider the function

$$f \colon [0, \infty] \longrightarrow [p/q, r/s] \qquad \text{given by} \qquad f(t) = \frac{p + tr}{q + ts}.$$

As t increases from 0 to ∞, f increases from p/q to r/s. Thus f is a bijection. Moreover it is clear that $f(t)$ is rational if and only if t is rational. Thus we may assume that $t = u/v$, where u, $v > 0$ and $(u, v) = 1$. We have

$$f\left(\frac{u}{v}\right) = \frac{vp + ur}{vq + us}.$$

As

$$q(vp+ur)-p(vq+us)=u(qr-ps)=u$$
$$s(vp+ur)-r(vq+us)=v(ps-qr)=-v,$$

it follows that $vp+ur$ is coprime to $vq+us$.

It follows that as u and v run over all coprime integers, $f(u/v)$ runs over all rational numbers between p/q and r/s. Amongst all such choices, $u=v=1$ gives the smallest denominator. $f(1)$ is the mediant of p/q and r/s and for future reference note that

$$|(p+r)q-(q+s)p|=1 \qquad \text{and} \qquad |r(q+s)-s(p+r)|=1.$$

Since $q+s>\max(q,s)$, (2) holds.

We now turn to (1). We prove this by induction on n. If $n=1$ then $p/q=a/1$ and $r/s=(a+1)/1$ so that

$$\begin{aligned}|ps-qr| &= |a\cdot 1-(a+1)\cdot 1| \\ &= 1.\end{aligned}$$

Thus (1) holds when $n=1$.

Now suppose that (1) holds for n. The only elements of $\mathcal{F}_{n+1}$ not in $\mathcal{F}_n$ are mediants of elements of $\mathcal{F}_n$ and we have already checked (1) in this case. Thus (1) holds by induction. □

Theorem 15.5 (Hurwitz)**.** *Suppose that the real number x is between two adjacent elements r/s and u/v of $\mathcal{F}_n$.*

Then at least one of the three numbers

$$\frac{r}{s} \qquad \frac{u}{v} \qquad \text{and} \qquad \frac{l}{m}=\frac{(r+u)}{(s+v)}$$

is a solution of the equation

$$\left|x-\frac{p}{q}\right|<\frac{1}{\sqrt{5}q^2}.$$

In particular if x is irrational then we may find infinitely many such p and q.

Proof. Possibly relabelling, we may assume that

$$\frac{r}{s}<\frac{l}{m}<\frac{u}{v}.$$

If p/q is one of these three numbers and c is a positive real number then let $I_c(p/q)$ be the interval

$$\left[\frac{p}{q}-\frac{1}{cq^2},\frac{p}{q}+\frac{1}{cq^2}\right].$$

We want to find the largest value of c so that the three intervals $I_c(r/s)$, $I_c(l/m)$ and $I_c(u/v)$ completely cover the interval

$$I = \left[\frac{r}{s}, \frac{u}{v}\right].$$

Note that $I_c(r/s)$ intersects $I_c(u/v)$ if

$$\frac{r}{s} + \frac{1}{cs^2} \geq \frac{u}{v} - \frac{1}{cv^2}.$$

Rearranging, this gives

$$\frac{1}{c}\left(\frac{1}{s^2} + \frac{1}{v^2}\right) \geq \frac{u}{v} - \frac{r}{s} = \frac{1}{vs},$$

so that

$$c \leq \frac{v}{s} + \frac{s}{v}.$$

If we let

$$f(t) = t + \frac{1}{t}$$

then

$$c \leq f\left(\frac{v}{s}\right).$$

By a similar analysis, $I_c(r/s)$ and $I_c(l/m)$ intersect if

$$c \leq f\left(\frac{m}{s}\right) = f\left(1 + \frac{v}{s}\right).$$

Consider the problem of trying to cover the left-hand portion

$$\left[\frac{r}{s}, \frac{l}{m}\right]$$

of the interval I by the union I_c of the three intervals. I is covered by I_c if either of these intervals intersect, that is, we are done if

$$c \leq \max\left(f\left(\frac{v}{s}\right), f\left(1 + \frac{v}{s}\right)\right).$$

So we are definitely done if

$$c \leq \min_{t>0} \max\left(f\left(t\right), f\left(1 + t\right)\right)$$

since we are taking a minimum over values that include

$$t = \frac{v}{s}.$$

The minimum occurs for that value t_0 of t for which $f(t) = f(1+t)$. This gives the equation

$$t + \frac{1}{t} = t + 1 + \frac{1}{1+t}.$$

Cancelling the t and cross-multiplying, it follows that

$$1 + t = t(1 + t) + t.$$

Thus

$$t^2 + t - 1 = 0.$$

The positive root of this equation is

$$t_0 = \frac{\sqrt{5} - 1}{2}.$$

It follows that

$$c_0 = \sqrt{5}.$$

By symmetry the right-hand portion

$$\left[\frac{l}{m}, \frac{u}{v}\right]$$

is also covered if $c \leq \sqrt{5}$.

Thus I belongs to the union I_c of the intervals if $c \leq \sqrt{5}$. This shows we get an inequality

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{\sqrt{5}q^2}.$$

As $\sqrt{5}$ is irrational, we must in fact have strict inequality.

If x is irrational the interval determined by adjacent points of $\mathcal{F}_n$ to which x belongs must shrink down to x, on both sides of x. Thus we get infinitely many p/q this way. □