14. Diophantine approximation

Given an irrational number x we have seen that it is possible to find infinitely many numbers p and q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Note that this gives a way to differentiate rationals from irrationals.

We will now study how to use approximation to differentiate algebraic numbers from transcendental numbers.

Definition 14.1. *A real number ξ is called **transcendental** if it is not algebraic.*

Theorem 14.2 (Liouville)**.** *If α is algebraic of degree $n > 1$ then there is a constant $c = c(\alpha) > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n}$$

for every pair of integers p and $q > 0$.

Proof. If p and q are such that

$$\left|\alpha - \frac{p}{q}\right| > 1,$$

then

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n},$$

for any $c \leq 1$. Therefore we may assume that

$$\left|\alpha - \frac{p}{q}\right| \leq 1.$$

In particular p/q is bounded.

Let $m_\alpha(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α. By assumption $m_\alpha(x)$ has degree n. Multiplying through to clear denominators and then dividing out any common factors, we may find an irreducible polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$$

such that $f(\alpha) = 0$.

As $f(x)$ is irreducible and $n > 1$, we have

$$f(p/q) \neq 0,$$

for $p/q \in \mathbb{Q}$. It follows that

$$\begin{aligned}\left|q^n f\left(\frac{p}{q}\right)\right| &= |a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_n q^n| \\ &\geq 1.\end{aligned}$$

The mean value theorem implies that

$$\begin{aligned}f\left(\frac{p}{q}\right) &= f\left(\frac{p}{q}\right) - f(\alpha) \\ &= \left(\frac{p}{q} - \alpha\right) f'(\xi)\end{aligned}$$

for some ξ between p/q and α. It follows that ξ is bounded, so that $|f'(\xi)|$ is bounded from above, say by c_1. There is no harm in assuming that $c_1 > 1$. It follows that if $c = 1/c_1$ then

$$\begin{aligned}\left|\alpha - \frac{p}{q}\right| &= \frac{|q^n f(p/q)|}{q^n f'(\xi)} \\ &\geq \frac{1}{c_1 q^n} \\ &= \frac{c}{q^n}.\end{aligned}$$

□

Corollary 14.3. *The number*

$$\xi = 1 \pm \frac{1}{2^{1!}} \pm \frac{1}{2^{2!}} \pm \frac{1}{2^{3!}} \pm \cdots \pm \frac{1}{2^{n!}} \pm \dots,$$

is transcendental, no matter how the signs are chosen.

In particular there are uncountably many transcendental numbers.

Proof. We are going to apply (14.2); it suffices to show that we can approximate ξ very well by rational numbers.

Fix once and for all a choice of signs.

Let $q_n = 2^{(n-1)!}$ and let

$$p_n = q_n(1 \pm 2^{-1!} \pm 2^{-2!} \pm 2^{-3!} \pm \cdots \pm 2^{-(n-1)!}).$$

Then p_n/q_n is a partial sum of the series for ξ and

$$\left|\xi - \frac{p_n}{q_n}\right| \leq 2^{-n!} + 2^{-(n+1)!} + \dots.$$

Note that

$$\begin{aligned}2^{(n+k)!} &\geq 2^{n!(n+k)} \\ &\geq (2^{n!})^k.\end{aligned}$$

Thus the series on the RHS is dominated by the geometric series

$$2^{-n!} + 2^{-n!} + (2^{-n!})^2 + (2^{-n!})^3 + \dots.$$

If we sum the geometric series we get

$$\begin{aligned} \left| \xi - \frac{p_n}{q_n} \right| &\leq 2^{-n!} + \frac{2^{-n!}}{1 - 2^{-n!}} \\ &\leq 2\frac{2^{-n!}}{1 - 2^{-n!}} \\ &\leq 4 \cdot 2^{-n!} \\ &= 4 \cdot q_n^{-n}. \end{aligned}$$

Thus ξ is not algebraic of degree $n > 1$ by (14.2). Note that ξ is not rational, as its base 2 expansion is not periodic, so that ξ is not algebraic of degree $n = 1$. Thus ξ is transcendental. □

Definition 14.4. *We say that a number ξ is **Liouville** if it fails the inequality* (14.2) *for every n.*

Let L denote the set of all Liouville numbers. (14.2) implies that every irrational Liouville number is transcendental. (14.3) implies that L is uncountable.

Curiously, in a completely different sense, it turns out that Liouville numbers are quite rare. Given a set $A \subset \mathbb{R}$, let

$$\chi_A \colon \mathbb{R} \longrightarrow \{ 0, 1 \}$$

be the indicator function of A, so that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then the area under the graph of L is zero (using the Lebesgue integral, not the Riemann integral; put differently L has measure zero), so that if one was to pick a real number at random the chance of picking a Liouville number is zero. By comparison the probability of choosing a transcendental number is one.