3. Chern classes

We have already seen that the first chern class gives a powerful way to connect line bundles, sections of line bundles and divisors. We want to generalise this to higher rank.

Given any vector bundle we can define higher chern classes. There are many ways to view chern classes, all of which are useful. We present two ways to look at them.

The first is topological. One can view chern classes as (partial) obstructions to the vector bundle being trivial. The first case is a line bundle. If a line bundle is trivial, that is, isomorphic to a product, then we can find a global non-vanishing section of the line bundle. One direction is clear, the trivial line bundle has the section $(x, 1)$, which is nowhere vanishing. On the other hand, if σ is a non-vanishing section of L then define a map

$$X \times \mathbb{C} \longrightarrow L$$

by sending (x, λ) to $\lambda\sigma(x)$.

Note that there are two different ways in which a vector bundle might be trivial. It might be topological trivial, that is, the isomorphism is only a continuous map. Or the isomorphism might be holomorphic. This reflects the two different types of first chern class, the topological first chern class and the more refined first chern class, which takes values in the space of Cartier divisors modulo linear equivalence. Either way, the first chern is defined by taking the equivalence class of the zero locus of a section. The fact that this equivalence class is non-zero means we cannot alter the section and make it nowhere vanishing, so that we get an obstruction to triviality.

Now suppose we have a vector bundle E of higher rank r. There is more than way to find obstructions to trivialising the bundle. Consider the problem of finding a nowhere zero section. We expect a section of a vector bundle of rank r to vanish in codimension r. Indeed, locally the vector bundle is trivial and a section of a vector bundle of rank r is a tuple of r holomorphic functions, which we expect to have a common zero in codimension r.

If L is an ample line bundle then results of Serre imply that $E \otimes L^k$ is globally generated for k sufficiently large and we can always find a section which vanishes in codimension r. We can then use linearity (more about this later) to define the rth chern class $c_r(E)$ of E. Topologically it takes values in $H^{2r}(X, \mathbb{Z})$ and there is a more refined version which takes values in the space of codimension r cycles, modulo rational equivalence.

At the other extreme, suppose the vector bundle were trivial. Then we could find r sections which fibre by fibre are a basis for each fibre. These r sections would then define a non-vanishing section of the highest wedge of E,

$$L = \bigwedge^r E.$$

Note that L is a line bundle, known as the determinant line bundle and we are simply asking if we can find a non-vanishing section of L, that is, we are asking if L is the trivial vector bundle. We define the first chern of E as the first chern class of L,

$$c_1(E) = c_1(\bigwedge^r E).$$

More generally still, the kth chern of a vector bundle E is a measure of how hard it is to find $k - r$ independent sections. The kth chern class of a vector bundle of rank r is a cycle that lives in either $H^{2k}(X, \mathbb{Z})$, of the space of codimension r cycles modulo rational equivalence.

To proceed further, it is convenient to introduce the second way to look at chern classes. This takes a more algebraic approach. We first bundle all of the chern classes together to get the total chern class

$$c(E) = c_0(E) + c_1(E) + c_2(E) + \cdots + c_r(E).$$

Grothendieck observed that the total chern class is unique, given the following axioms:

(1) $c_0(E) = 1$, the class of X.

(2) $c_1(\mathcal{O}_X(D)) = [D]$.

(3) If $f\colon Y \longrightarrow X$ is a morphism then

$$f^*(c(E)) = c(f^*E).$$

(4) If

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

is a short exact sequence of locally free sheaves then

$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}).$$

As a baby case of (4), note that if E_1 and E_2 are vector bundles then

$$c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2).$$

In fact, here is how to define the chern classes, using these properties. Given the vector bundle E, let $Y = \mathbb{P}(E)$ be the associated projective bundle. Fibre by fibre, $\pi\colon Y \longrightarrow X$ is a family of projective spaces $\mathbb{P}^{r-1}$. The cohomology of $\mathbb{P}^{r-1}$ is

$$\frac{\mathbb{Z}[x]}{\langle x^r \rangle},$$

where x is in degree 2, the class of a hyperplane. The universal line bundle $\mathcal{O}_Y(1)$ restricts to a line bundle whose first chern class is x. So the first chern class ξ of $\mathcal{O}_Y(1)$ restricts to the generator x on each fibre. Consider the first $r+1$ powers of ξ. Some linear combination of these sums to zero in the cohomology of Y,

$$\xi^r - c_1\xi^{r-1} + c_2\xi^{r-2} - \cdots + (-1)^r c_r(E) = 0.$$

Let's compute the chern classes of the tangent bundle. We have the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \longrightarrow T_X \longrightarrow 0.$$

It follows that

$$c(\mathcal{O}_{\mathbb{P}^n})c(T_X) = c(\mathcal{O}_{\mathbb{P}^n}^{n+1}(1)).$$

Now the total chern class of a trivial line bundle is trivial

$$c(\mathcal{O}_{\mathbb{P}^n}) = 1 \qquad \text{and} \qquad c(\mathcal{O}_{\mathbb{P}^n}(1)) = 1 + H,$$

where H is the class of a hyperplane. Thus

$$\begin{aligned} c(T_X) &= c(\mathcal{O}_{\mathbb{P}^n}^{n+1}(1)) \\ &= \prod_{i=0}^{n}(1+H) \\ &= (1+H)^{n+1} \\ &= 1 + (n+1)H + \frac{(n+1)n}{2}H^2 + \dots. \end{aligned}$$

Consider what happens on $\mathbb{P}^2$. The tangent bundle has rank two. Its total chern class is

$$1 + 3H + 3H^2.$$

(Note that our computation of the second chern class is consistent with Gauss-Bonnett, since the topological Euler characteristic is indeed $3 = 1 + 1 + 1$).

If it were isomorphic to a direct sum then its chern classes would be

$$\begin{aligned} c(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)) &= c(\mathcal{O}_{\mathbb{P}^2}(a))c(\mathcal{O}_{\mathbb{P}^2}(b)) \\ &= (1+aH)(1+bH) \\ &= 1 + (a+b)H + abH^2. \end{aligned}$$

Thus $a + b = 3$ and $ab = 3$. But this is not possible for integers. Thus the tangent bundle does not split.

The chern classes of a vector bundle provide a useful way to chart out the territory of all vector bundles.

Splitting principle One can use (4) to compute chern classes in many situations. If we want to compute some chern classes, in most cases we can pullback to a situation where the vector bundle splits and the pullback map is injective. Thus in many cases we can compute as though the vector bundles splits.

One can use the splitting principle to compute the chern classes of tensor products.

Question 3.1. *What are the chern classes of the tensor product of a vector bundle and a line bundle?*

Suppose the vector bundle is E and the line bundle is L. We want compute to

$$c(E \otimes L).$$

We use the splitting principle. Assume that

$$E = L_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r.$$

Then

$$\begin{aligned} c(E) &= c(L_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r) \\ &= \prod_{i=1}^{r} c(L_i) \\ &= \prod_{i=1}^{r} (1 + \alpha_i), \end{aligned}$$

where $\alpha_i = c_1(L_i)$. Suppose that

$$c_1(L) = \beta.$$

Then

$$\begin{aligned} c(E \otimes L) &= c((L_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r) \otimes L) \\ &= c((L_1 \otimes L) \oplus (L_2 \otimes L) \oplus (L_3 \otimes L) \oplus \cdots \oplus (L_r \otimes L)) \\ &= \prod_{i=1}^{r} c(L_i \otimes L) \\ &= \prod_{i=1}^{r} (1 + \alpha_i + \beta) \\ &= 1 + (\sum \alpha_i + r\beta) + (\sum_{i \neq j} \alpha_i \alpha_j + (r-1)\beta(\sum_i \alpha_i) + \binom{r}{2}\beta^2) + \dots \\ &= 1 + c_1(E) + rc_1(L) + c_2(E) + (r-1)c_1(E)c_1(L) + \binom{r}{2} c_1^2(L) + \dots \end{aligned}$$

Since the formula for the tensor product is actually quite involved, it is natural to exponentiate to get a simpler formula. Formally, if

$$c(E) = \prod_{i=1}^{r}(1 + \alpha_i),$$

so that the chern classes of E are the symmetric functions in $\alpha_1, \alpha_2, \dots, \alpha_r$ then the chern character is

$$\operatorname{ch}(E) = \sum_{i=1}^{r} e^{\alpha_i},$$

where we use the usual formula for the exponential. Note that then chern character is additive on exact sequences and multiplicative on tensor products.

$$\begin{aligned}\operatorname{ch}(E_1 \oplus E_2) &= \operatorname{ch}(E_1) + \operatorname{ch}(E_2)\\ \operatorname{ch}(E_1 \otimes E_2) &= \operatorname{ch}(E_1)\operatorname{ch}(E_2).\end{aligned}$$

The first few terms of the chern character are

$$\operatorname{ch}(E) = r + c_1(E) + \frac{1}{2}(c_1^2 - c_2) + \dots$$