17. Extremal Bundles

We look at various extremal cases of Hartshorne's conjecture. Let $Y \subset \mathbb{P}^{m+2}$ be a local complete intersection of codimension two of degree d such that $\det N_{Y/\mathbb{P}^{m+2}} = \mathcal{O}_Y(k)$. We have already seen that if

$$k \geq \frac{d}{\mu} + \mu,$$

for some $\mu \in (0, m]$ then Y is a complete intersection.

If E is the associated vector bundle then

$$c_1(E) = k$$
$$c_2(E) = d.$$

Suppose that α and β are the chern roots, so that

$$c(E) = (1 + \alpha)(1 + \beta)$$

and $k = c_1(E) = \alpha + \beta$ and $d = c_2(E) = \alpha\beta$.

Suppose that one of α and β lies in the interval $(0, m]$. Then (1) holds, with $\mu = \alpha$, so that Y is a complete intersection. Note that $\alpha > 0$ and $\beta > 0$ since $\alpha\beta = d > 0$ and $\alpha + \beta = k > 0$. Therefore if

$$k < 2m$$

then one of α and β belongs to $(0, m]$,

Suppose that $k > 2m$ and the first inequality does not hold. Then

$$\frac{d}{\mu} + \mu > 2m.$$

Thus

$$d > 2m\mu - \mu^2.$$

The RHS is maximised if we take $\mu = m$, in which case

$$d > m^2.$$

It follows that if $d \leq m^2$ then Y is a complete intersection.