16. Stable vs unstable

Vector bundles are naturally divided into two quite distinct types, stable and unstable. We will only scrape the surface of this important topic.

Definition 16.1. *Let $\mathcal{E}$ be a coherent sheaf on a quasi-projective variety. We say that $\mathcal{E}$ is* ***reflexive*** *if it isomorphic to its double dual.*

It is not hard to see that the double dual of any coherent sheaf is reflexive. The main technical consequence of some results in homological algebra of local rings we will need is the following:

Lemma 16.2. *A rank one sheaf on a smooth variety is reflexive if and only if it is a line bundle.*

Using (16.2) we can define the first chern class of a torsion free sheaf $\mathcal{E}$ by the rule

$$c_1(\mathcal{E}) = c_1((\det \mathcal{E})^{**}),$$

where the determinant just means take the highest wedge.

Definition 16.3. *Let $\mathcal{E}$ a torsion free sheaf of rank r on a projective variety X and let H be an ample divisor. The* ***slope*** *of $\mathcal{E}$ with respect to H, denoted $\mu(\mathcal{E})$, is the ratio*

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{r}.$$

We say that $\mathcal{E}$ is ***semistable*** *if the slope of any coherent subsheaf $\mathcal{F}$ is at most the slope of $\mathcal{E}$,*

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

We say that $\mathcal{E}$ is ***stable*** *if we always have strict inequality, when the rank of $\mathcal{F}$ is neither zero nor r. We say that $\mathcal{E}$ is* ***unstable*** *if it is not stable.*

Note that the definition of stability might change if we change H but it won't change if we replace H by a multiple. In the case of $\mathbb{P}^n$ there is therefore no ambiguity in dropping the reference to H.

Theorem 16.4. *Let $\mathcal{E}$ be a torsion free sheaf on $\mathbb{P}^n$.*

TFAE

1. *$\mathcal{E}$ is stable (respectively semistable).*
2. *$\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively $\leq$) for all coherent subsheaves (whose rank is neither zero nor r) such that $\mathcal{E}/\mathcal{F}$ is torsion free.*
3. *$\mu(\mathcal{Q}) > \mu(\mathcal{E})$ (respectively $\geq$) for all torsion free quotient sheaves (whose rank is neither zero nor r).*

Proof. (1) clearly implies (2). Suppose that $\mathcal{F} \subset \mathcal{E}$ is a subsheaf. Let

$$\mathcal{Q} = \frac{\mathcal{E}}{\mathcal{F}}.$$

and let

$$\mathcal{Q}'$$

be the free part of $\mathcal{Q}$. Then $\mathcal{Q}'$ is torsion free and there is a natural surjective map $\mathcal{E} \longrightarrow \mathcal{Q}'$. If $\mathcal{H}$ is the kernel then $\mathcal{F} \subset \mathcal{H}$ and both sheaves have the same rank. As $c_1(\mathcal{F}) \leq c_1(\mathcal{H})$ we have

$$\mu(\mathcal{F}) \leq \mu(\mathcal{H}).$$

Thus (2) implies (1).

Now suppose that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

is a short exact sequence of torsion free sheaves, or ranks s, r and t so that $r = s + t$. Note that

$$c_1(\mathcal{E}) = c_1(\mathcal{F}) + c_1(\mathcal{Q}).$$

We have

$$\mu(\mathcal{F}) < \mu(\mathcal{E})$$

if and only if

$$\frac{s+t}{s} c_1(\mathcal{F}) < c_1(\mathcal{F}) + c_1(\mathcal{Q}),$$

if and only if

$$c_1(\mathcal{F}) < \frac{s}{t} c_1(\mathcal{Q}),$$

if and only if

$$c_1(\mathcal{Q}) + c_1(\mathcal{F}) < \frac{s+t}{t} c_1(\mathcal{Q})$$

if and only if

$$\mu(\mathcal{Q}) > \mu(\mathcal{E}).$$

□

Lemma 16.5.

(1) Line bundles are stable.
(2) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are torsion free sheaves than $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semistable if and only if $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable with the same slope.
(3) $\mathcal{E}$ is semistable if and only if $\mathcal{E}^*$ is semistable.
(4) If $\mathcal{E}$ is semistable then $\mathcal{E}(k)$ is semistable for all $k \in \mathbb{Z}$.

Proof. (1) is trivial.

Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable with the same slope $\mu = \mu(\mathcal{E}_i)$. Then $\mu = \mu(\mathcal{E}_1 \oplus \mathcal{E}_2)$. Suppose that $\mathcal{F} \subset \mathcal{E}_1 \oplus \mathcal{E}_2$ is a coherent subsheaf. Then there is an induced commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}_2 \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{F}_1 = \mathcal{F} \cap (\mathcal{E}_1 \oplus 0)$ and $\mathcal{F}_2 = \mathcal{F} \cap (0 \oplus \mathcal{E}_2)$. As $\mathcal{E}_i$ is semistable it follows that $c_1(\mathcal{F}_i) \leq \mu s_i$, where s_i is the rank of $\mathcal{F}_i$.

It follows that

$$\begin{aligned} \mu(\mathcal{F}) &= \frac{c_1(\mathcal{F}_1) + c_1(\mathcal{F}_2)}{s_1 + s_2} \\ &\leq \mu \frac{s_1 + s_2}{s_1 + s_2} \\ &= \mu. \end{aligned}$$

Thus $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semistable.

Conversely if $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semistable then $\mu(\mathcal{E}_i) = \mu$ as $\mathcal{E}_i$ is both a sub and a quotient sheaf. If $\mathcal{E}_1$ is not semistable then let $\mathcal{F}_1$ be a destabilising subsheaf of rank s. Consider

$$\mathcal{F}_1 \oplus \mathcal{E}_2 \subset \mathcal{E}_1 \oplus \mathcal{E}_2.$$

Then

$$\begin{aligned} \mu(\mathcal{F}_1 \oplus \mathcal{E}_2) &= \frac{c_1(\mathcal{F}_1) + c_1(\mathcal{E}_2)}{s + r_2} \\ &> \frac{\mu s + \mu r_2}{s + r_2} \\ &= \mu. \end{aligned}$$

Thus (2) holds.

Note that if $\mathcal{E}$ is semistable then $\mathcal{E}^*$ by (16.4). Thus (3) holds.

Note that if $\mathcal{F} \subset \mathcal{E}$ then $\mathcal{F}(k) \subset \mathcal{E}(k)$. As the slope of $\mathcal{E}$ and $\mathcal{E}(k)$ are the same, (4) is clear. $\square$

Definition 16.6. *Let E be a vector bundle of rank r on $\mathbb{P}^n$.*

*We say that E is **normalised** if $-r < c_1(E) \leq 0$.*

It is clear that if E is a vector bundle then there is a unique integer k so that $E(k)$ is normalised.

Lemma 16.7. *Let E be a rank two normalised vector bundle on $\mathbb{P}^n$.*

Then E is stable if and only if $h^0(\mathbb{P}^n, E) = 0$.

If $c_1(E)$ is even then E is semistable if and only if $h^0(\mathbb{P}^n, E(-1)) = 0$.

Proof. We prove the first statement. One direction is clear; if $h^0(\mathbb{P}^n, E) \neq 0$ then $\mathcal{O}_{\mathbb{P}^n}$ is a torsion free subsheaf of E. The slope of both E and $\mathcal{O}_{\mathbb{P}^n}$ is zero and so E is not stable.

Now suppose that $h^0(\mathbb{P}^n, E) = 0$. Suppose that $\mathcal{F} \subset E$ is a torsion free subsheaf of rank one. If we replace $\mathcal{F}$ by its double dual then the slope only goes up. Thus we may assume that $\mathcal{F}$ is reflexive, that is, we may assume that $L = \mathcal{F}$ is a line bundle. If $L \simeq \mathcal{O}_{\mathbb{P}^n}(k)$ then $k < 0$ as $h^0(\mathbb{P}^n, E) = 0$.

But then

$$\begin{aligned}\mu(L) &\leq -1 \\ &< -\frac{1}{2} \\ &= \mu(E) = 0.\end{aligned}$$

Thus E is stable.

We now turn the second statement. One direction is again clear; if $h^0(\mathbb{P}^n, E(-1)) \neq 0$ then $\mathcal{O}_{\mathbb{P}^n}(1)$ is a torsion free subsheaf of E. The slope of E is zero and of $\mathcal{O}_{\mathbb{P}^n}(1)$ is one and so E is not semistable.

Now suppose that $h^0(\mathbb{P}^n, E(-1)) = 0$. Suppose that $\mathcal{F} \subset E$ is a torsion free of rank one. As before we may assume that $L = \mathcal{F}$ is a line bundle. If $L \simeq \mathcal{O}_{\mathbb{P}^n}(k)$ then $k < 1$ as $h^0(\mathbb{P}^n, E(-1)) = 0$.

But then

$$\mu(L) \leq 0 \leq \mu(E) = 0.$$

Thus E is semistable. $\square$

Lemma 16.8. *Let E be a rank two torsion free sheaf on $\mathbb{P}^2$ with chern classes c_1 and c_2.*

If E is stable then

$$c_1^2 - 4c_2 < 0.$$

If E is semistable then

$$c_1^2 - 4c_2 \leq 0.$$

Proof. The discriminant

$$\Delta = c_1^2 - 4c_2$$

is invariant under twisting as is stability. Thus we may assume that E is normalised.

Suppose that E is stable. Then

$$H^0(\mathbb{P}^2, E) = 0$$

and by duality

$$H^2(\mathbb{P}^2, E) = 0.$$

Hence
$$\chi(\mathbb{P}^2, E) = -h^1(\mathbb{P}^2, E) \leq 0.$$
Now Riemann-Roch for E on $\mathbb{P}^2$ reads
$$\chi(\mathbb{P}^2, E) = \frac{1}{2}\left(c_1^2 - 2c_2 + 3c_1 + 4\right).$$

Indeed, the Riemann-Roch formula is a rational polynomial in the chern classes of E. If we consider what happens for $E = \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ we get
$$\begin{aligned}\chi(\mathbb{P}^2, E) &= \binom{a+2}{2} + \binom{b+2}{2} \\ &= \frac{1}{2}\left((a+b)^2 - 2ab + 3(a+b) + 4\right) \\ &= \frac{1}{2}\left(c_1^2 - 2c_2 + 3c_1 + 4\right)\end{aligned}$$
and this determines the formula.

Thus we have
$$c_1^2 - 2c_2 + 3c_1 + 4 \leq 0.$$
There are two cases. If $c_1 = 0$ then $-2c_2 + 4 \leq 0$ so that $c_2 \geq 2$ and so $\Delta \leq 0 - 8 < 0$. If $c_1 = -1$ then $-2c_2 + 2 \leq 0$ so that $c_2 \geq 1$. In this case $\Delta \leq 1^2 - 4 < 0$.

Now suppose that E is semistable but not stable. Then c_1 is even so that we may assume that $c_1 = 0$. But then
$$\begin{aligned}0 &\leq h^1(\mathbb{P}^2, E(-1)) \\ &= -\chi(\mathbb{P}^2, E(-1)) \\ &= -\frac{1}{2}\left(2^2 - 2(c_2 + 1) - 6 + 4\right) \\ &= c_2(E)\end{aligned}$$
Thus $c_2(E) \geq 0$ so that $\Delta \leq 0$. □

We end with the connection between stability and simplicity.

Lemma 16.9. *Let $\phi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ be a non-trivial sheaf map between semistable sheaves of the same slope.*

If one of the sheaves is stable then ϕ is a monomorphism or generically an epimorphism.

Proof. Let $\mathcal{I} = \operatorname{Im}\phi$ be the image of ϕ. Then I is a torsion free subsheaf of rank at least one, as ϕ is non-trivial.

Suppose that
$$\operatorname{rk}(\mathcal{I}) < \operatorname{rk}(\mathcal{E}_1) \qquad \text{and} \qquad \operatorname{rk}(\mathcal{I}) < \operatorname{rk}(\mathcal{E}_2).$$

If $\mathcal{E}_1$ is stable then

$$\begin{aligned}\mu(\mathcal{I}) &\leq \mu(\mathcal{E}_2)\\ &< \mu(\mathcal{I})\end{aligned}$$

and if $\mathcal{E}_2$ is stable then

$$\begin{aligned}\mu(\mathcal{I}) &< \mu(\mathcal{E}_2)\\ &\leq \mu(\mathcal{I}),\end{aligned}$$

which are both impossible.

Therefore, either $\mathrm{rk}(\mathcal{I}) = \mathrm{rk}(\mathcal{E}_1)$, in which case ϕ is a monomorphism or $\mathrm{rk}(\mathcal{I}) = \mathrm{rk}(\mathcal{E}_2)$, in which case ϕ is generically an epimorphism. □

Corollary 16.10. *Let $\phi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ be a non-trivial sheaf map between semistable sheaves with the same rank and first chern class.*

If one of the sheaves is stable then ϕ is an isomorphism.

Proof. By (16.9) ϕ is a monomorphism and so

$$\det\phi\colon\ \det\mathcal{E}_1 \longrightarrow \det\mathcal{E}_2$$

is also a monomorphism. As the first chern classes are the same, it follows that $\det\phi$ is an isomorphism so that ϕ is an isomorphism. □

Theorem 16.11. *Stable bundles are simple.*

Proof. Let $\phi\colon E \longrightarrow E$ be an endomorphism of a stable bundle.

Pick a point $x \in \mathbb{P}^n$. Then $\phi_x\colon E_x \longrightarrow E_x$ is a linear endomorphism and so it has an eigenvalue λ. It follows that $\phi - \lambda\mathrm{id}_E$ is not an isomorphism so that is must be the zero map. But then ϕ is a homotherty so that E is simple. □

Theorem 16.12. *Every simple rank two vector bundle on $\mathbb{P}^n$ is stable.*

Proof. We may assume that E is normalised. If E is not stable then

$$h^0(\mathbb{P}^n, E) \neq 0$$

so that

$$h^0(\mathbb{P}^n, E^*) \neq 0$$

as $E^* \simeq E \otimes \det E^*$. But then E is not simple. □