13. Topological versus holomorphic classification I

It is interesting to compare the topological versus the holomorphic classification of vector bundles on $\mathbb{P}^n$. Of course every holomorphic vector bundle gives rise to a topological vector bundle. A priori, topological vector bundle means that the transition functions are continuous but it is not hard to show that we can smooth the transition functions and that two smooth vector bundles that are topologically equivalent are isomorphic as smooth vector bundles.

We first review the topological classification. Most of the time we will not provide any proofs. Recall that topological line bundles on $\mathbb{P}^n$ are classified by their first chern class c_1, which is an integer. Thus the topological and the holomorphic classification coincide for line bundles.

We already showed that topological vector bundles on $\mathbb{P}^1$ are classified by their first chern class. Holomorphic vector bundles or rank r are classified by a decreasing sequence of integers

$$a_1 \geq a_2 \geq \cdots \geq a_r.$$

Two such are topologically equivalent if

$$\sum a_i = \sum b_i.$$

Schwarzenberger showed that the chern classes of holomorphic vector bundles must satisfy various congruences, as a consequence of the (Hirzebruch)-Riemann-Roch theorem. On the other hand, if one applies the Atiyah-Singer index theorem the Schwarzenberger conditions also hold for any topological vector bundle.

Suppose that we encode the chern classes as a polynomial with integer coefficients

$$c_t(E) = 1 + c_1(E)t + \cdots + c_r(E)t^r \in \mathbb{Z}[t].$$

$c_t(E)$ is called the **chern polynomial** of E. Suppose that we factor this polynomial over the complex numbers

$$c_t(E) = \prod_{i=1}^{r}(1 + x_i t).$$

The numbers $c_1, c_2, \dots, c_r$ must then satisfy the **Schwarzenberger condition**,

$$(S^r_n) \qquad \sum_{i=1}^{r} \binom{n + x_i + s}{s} \in \mathbb{Z} \qquad \text{for every} \qquad s \in \mathbb{Z}.$$

It is straightforward to calculate what these conditions reduce to for low values of r and n. S^1_n and S^2_2 impose no conditions at all. S^2_3 is equivalent to

$$c_1 c_2 \equiv 0 \mod 2.$$

S_4^2 is equivalent to

$$c_2(c_2 + 1 - 3c_1 - 2c_1^2) \equiv 0 \mod 12.$$

Finally, S_3^3 is equivalent to

$$c_1 c_2 \equiv c_3 \mod 2.$$

Note that if E is a rank r bundle over $\mathbb{P}^n$ and $r \geq n$ then E has $n-r$ linearly independent sections. These linearly independent sections define a sub vector bundle of rank $n-r$ and this bundle splits off as a direct summand (using partitions of unity). Thus

$$E \simeq E' \oplus (\mathbb{P}^n \times \mathbb{C}^{n-4})$$

where E' is a vector bundle of rank n.

Topological vector bundles are completely classified. There is one bundle for each collection of chern classes which satisfy the Schwarzenberger conditions.

So for $n = 2$ the topological classification of rank two vector bundles corresponds to pairs of integers (c_1, c_2). For $n = 3$ the classification of rank three vector bundles corresponds to triples of integers (c_1, c_2, c_3) subject to $c_1 c_2 \equiv c_3 \mod 2$.

Now consider rank two vector bundles over $\mathbb{P}^3$. The Schwarzenberger condition is

$$c_1 c_2 \equiv 0 \mod 2.$$

It is known that there always a rank two vector bundle with these chern classes. If c_1 is odd there is one and if c_1 is even the are two topologically inequivalent bundles. These two bundles are distinguished by the α-invariant. If $c_1(E) = 2k$ then $c_1(E(-k)) = 0$. In this case the structure group can be reduced to $\mathrm{Sp}(1) \subset U(2)$.

Thus we just have to classify the symplectic line bundles on $\mathbb{P}^3$. Symplectic line bundles are topologically stable, classified by the group $\mathbb{Z} \oplus \mathbb{Z}_2$. Let π be the projection onto the second factor. The α invariant of a vector bundle E such that $c_1(E) = 2k$ is then given by

$$\alpha(E) = \pi(E(-k)) \in \mathbb{Z}_2.$$

If E is a holomorphic rank two vector bundle on $\mathbb{P}^3$ with $c_1(E) = 2k$ then

$$\alpha(E) \equiv h^0(\mathbb{P}^3, E(-k-2)) + h^1(\mathbb{P}^3, E(-k-2)) \mod 2.$$

Let us turn to the problem of which topological vector bundles have a holomorphic structure. We first treat the case of rank two on $\mathbb{P}^2$.

Recall that there is one topological vector bundle for every pair of integers (c_1, c_2).

Theorem 13.1 (Schwarzenberger)**.** *For every pair of integers $(c_1, c_2) \in \mathbb{Z}^2$ there is a holomorphic vector bundle E or rank two on $\mathbb{P}^2$.*

Proof. Let $\pi \colon X \longrightarrow \mathbb{P}^2$ blow up four points x_1, x_2, x_3 and x_4 of $\mathbb{P}^2$. Let E_i be the exceptional divisor over x_i and let

$$L = \mathcal{O}_X(D)$$

be the line bundle associated to the divisor $D \sum k_i E_i \geq 0$.

We consider extensions

$$0 \longrightarrow L \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(b) \longrightarrow V' \longrightarrow L^* \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

If $V' = \pi^* V$ for some rank two vector bundle V then V has chern classes

$$\begin{aligned} c_1(V) &= a + b \\ c_2(V) &= c_2(\pi^* V) \\ &= (D + b\pi^* H)(-D + a\pi^* H) \\ &= \sum k_i^2 + ab, \end{aligned}$$

where we used the fact that

$$E_i \cdot E_j = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

and $E_i \cdot \pi^* H = 0$.

As every positive integer is the sum of four squares, note that we can always choose a, b, $k_1 \geq 0$, $k_2 \geq 0$, $k_3 \geq 0$ and $k_4 \geq 0$ such that

$$\begin{aligned} c_1 &= a + b \\ c_2 &= \sum k_i^2 + a + b, \end{aligned}$$

where $a - b < 0$.

Proceeding in a similar manner to before, one can check that a vector bundle V' given as an extension

$$0 \longrightarrow L \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(b) \longrightarrow V' \longrightarrow L^* \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

is the pullback of a vector bundle V from $\mathbb{P}^2$ if and only if its restriction to any of the exceptional curves E_i is of the form

$$0 \longrightarrow \mathcal{O}_{E_i}(-k_i) \longrightarrow \mathcal{O}_{E_i}^{\oplus 2} \longrightarrow \mathcal{O}_{E_i}(-k_i) \longrightarrow 0.$$

(Recall that one shows that there is only one such extension over a neighbourhood of each exceptional.)

We are thus reduced to

Claim 13.2. *For a and $b \in \mathbb{Z}$, $a-b<0$ and $k_i \geq 0$, $i=1$, 2, 3 and 4 there is an extension*

$$0 \longrightarrow L \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(b) \longrightarrow V' \longrightarrow L^* \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

with

$$V'|_{E_i} \simeq \mathcal{O}_{E_i}^{\oplus 2}.$$

Proof of (13.2). We may assume that $b=0$. In this case, extensions of the form

$$0 \longrightarrow L \longrightarrow V' \longrightarrow L^* \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

are classified by

$$\operatorname{Ext}^1_X(L^* \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a), L) = H^1(X, L^{\otimes 2} \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)).$$

Let $E = E_1 + E_2 + E_3 + E_4$. Since there are extensions

$$0 \longrightarrow \mathcal{O}_E(D) \longrightarrow \mathcal{O}_E^{\oplus 2} \longrightarrow \mathcal{O}_E(-D) \longrightarrow 0,$$

it suffices to show that

$$H^1(X, L^{\otimes 2} \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)) \longrightarrow H^1(E, L^{\otimes 2})$$

is surjective. It suffices to show that

$$H^2(X, \mathcal{O}_X(2D-E) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)) = 0$$

As

$$K_X = \pi^*K_{\mathbb{P}^2} + E,$$

it follows that

$$\begin{aligned} h^2(X, \mathcal{O}_X(2D-E) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)) &= h^0(X, \mathcal{O}_X(2E-2D) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a-3)) \\ &\leq h^0(X, \mathcal{O}_X(2E) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a-3)) \end{aligned}$$

If we take the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X((k-1)E) \longrightarrow \mathcal{O}_X(kE) \longrightarrow \mathcal{O}_E(kE) \longrightarrow 0,$$

we see that

$$h^0(X, \mathcal{O}_X(kE) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a-3)) = h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^2}(a-3)),$$

for all $k \geq 0$. But

$$h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^2}(a-3)) \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a-3)) = 0,$$

as $a<0$. $\square$

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