11. The Serre construction

Suppose we are given a globally generated rank two vector bundle E on $\mathbb{P}^n$. Then the general global section σ of E vanishes in codimension two on a smooth subvariety Y. If E is decomposable then $\sigma = (F, G)$ where F and G are homogeneous polynomials, so that Y is the zero locus of F and G, a complete intersection. In fact we just need to know that Y has codimension two, in which case it has local complete intersection singularities, for this to work.

We want to reverse this process. Given a subvariety Y, with local complete intersection singularities, we want to construct a vector bundle E on Y and a global section which vanishes on Y. The idea is to extend the normal bundle (as Y is a lci, the normal sheaf is locally free) to a rank two vector bundle on $\mathbb{P}^n$.

Suppose we are given a rank two vector bundle E and a section $\sigma \in H^0(\mathbb{P}^n, E)$. We suppose that the zero locus Y of σ has codimension two. Locally E is trivial. If U is an open subset over which E is trivial, then σ corresponds to pair of regular functions f and g. It follows that Y is a local complete intersection and the ideal sheaf $\mathcal{I}_Y$ of Y is locally generated by f and g. In this case the conormal sheaf

$$N_Y^* = \frac{I_Y}{I_Y^2},$$

is locally free, with local generators f and g. Note that Y need not even be reduced.

Now we can write down a free resolution of the ideal sheaf on U.

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_U \oplus \mathcal{O}_U \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

If the first map is α and the second β then we have

$$\alpha(r) = (-fr, gr) \qquad \text{and} \qquad \beta(s, t) = fs + gt.$$

It is easy to check this sequence is exact, since we can check it is exact on stalks and use the fact that f and g is a regular sequence.

We can globalise to the following short exact sequence

$$0 \longrightarrow \det E^* \longrightarrow E^* \longrightarrow \mathcal{I}_Y \longrightarrow 0,$$

where

$$\alpha(\phi_1 \wedge \phi_2) = \phi_1(\sigma)\phi_2 - \phi_2(\sigma)\phi_1 \qquad \text{and} \qquad \beta(\phi) = \phi(\sigma).$$

This sequence is called the **Koszul complex** for σ. If Y has codimension two then it gives a global resolution of $\mathcal{I}_Y$ by locally free sheaves.

If we restrict this exact sequence to Y we get

$$(\det E^*)|_Y \longrightarrow E^*|_Y \longrightarrow \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \longrightarrow 0.$$

Note that the first map is in fact the zero map, as can be checked locally. It follows that we get an isomorphism

$$E^*|_Y \simeq \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2}.$$

Theorem 11.1 (Serre)**.** *Let Y be a local complete intersection of codimension two in $\mathbb{P}^n$. Suppose that the determinant of the normal bundle is the restriction of a line bundle on $\mathbb{P}^n$,*

$$\det N_{Y/\mathbb{P}^n} \simeq \mathcal{O}_Y(k) \qquad \textit{for some} \qquad k \in \mathbb{Z}.$$

Then there is a rank two vector bundle E on $\mathbb{P}^n$ a global section σ with zero locus Y and there is an exact sequence induced by σ

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0.$$

The chern classes of Y are given by

$$c_1(E) = k$$
$$c_2(E) = \deg Y.$$

Proof. If there is a bundle with these properties then

$$\det N^*_{Y/\mathbb{P}^n} = \det E^*|_Y.$$

In this case

$$\det E^* = \mathcal{O}_{\mathbb{P}^n}(-k)$$

so that

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-k) \longrightarrow E^* \longrightarrow \mathcal{J}_Y \longrightarrow 0.$$

Now extensions of $\mathcal{J}_Y$ by $\mathcal{O}_{\mathbb{P}^n}(-k)$ are controlled by

$$\operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{J}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

There is a spectral sequence who E_2-term is

$$E_2^{p,q} = H^p(\mathbb{P}^n, \mathbf{Ext}^q_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)))$$

and whose E_∞-term is

$$E_\infty^{p+q} = \operatorname{Ext}^{p+q}_{\mathbb{P}^n}(\mathcal{J}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

Chasing the spectral sequence we get an exact sequence

$$0 \longrightarrow H^1(\mathbb{P}^n, \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \longrightarrow \operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{J}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow$$
$$H^0(\mathbb{P}^n, \mathbf{Ext}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \longrightarrow H^2(\mathbb{P}^n, \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))).$$

On the other hand, the short exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

gives to long exact sequence of ext,

$$\begin{aligned}0 \longrightarrow \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \\ \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))).\end{aligned}$$

Since Y is a local complete intersection, it follows that it is Cohen-Macaulay. Therefore

$$\mathbf{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) = 0,$$

for $i = 0$ and 1. Thus

$$\begin{aligned}\mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) &= \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-k)) \\ &= \mathcal{O}_{\mathbb{P}^n}(-k).\end{aligned}$$

Thus the long exact sequence we got from the spectral sequence becomes

$$\begin{aligned}0 \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \\ H^0(\mathbb{P}^n, \mathbf{Ext}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \longrightarrow H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-k)).\end{aligned}$$

In particular, if $n > 3$ or $n = 2$ and $k < 3$ then

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq H^0(\mathbb{P}^n, \mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))).$$

Otherwise, we just get an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^2}(-k)) \longrightarrow H^0(\mathbb{P}^n, \mathbf{Ext}_{\mathbb{P}^2}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^2}(-k))) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-k)).$$

We turn to calculating

$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

From the long exact sequence associated to

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

we get

$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq \mathbf{Ext}^2_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

As Y is a local complete intersection of codimension two, we have the local fundamental isomorphism

$$\mathbf{Ext}^2_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\det \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2}, \mathcal{O}_Y(-k)).$$

By assumption

$$\frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \simeq \mathcal{O}_Y(-k),$$

so that

$$\mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\det \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2}, \mathcal{O}_Y(-k)) \simeq \mathcal{O}_Y.$$

Putting all of this together we have

$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq \mathcal{O}_Y.$$

It follows that

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq H^0(\mathbb{P}^n, \mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \simeq H^0(\mathcal{O}_Y, \mathcal{O}_Y).$$

Let F be the extension corresponding to $1 \in H^0(Y, \mathcal{O}_Y)$,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-k) \longrightarrow F \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

Then F is a coherent sheaf.

Claim 11.2. *F is a locally free sheaf.*

Proof of (11.2). Pick $x \in \mathbb{P}^n$. Then the image 1_x of 1 in $\mathcal{O}_{\mathbb{P}^n,x}$ lives in

$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))_x = \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{I}_{Y,x}, \mathcal{O}_{\mathbb{P}^n,x}(-k)).$$

This defines the extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n,x}(-k) \longrightarrow F_x \longrightarrow \mathcal{I}_{Y,x} \longrightarrow 0.$$

Since the 1_x generates the $\mathcal{O}_{\mathbb{P}^n,x}$-module

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{I}_{Y,x}, \mathcal{O}_{\mathbb{P}^n,x}(-k)) \simeq \mathcal{O}_{Y,x},$$

it follows that F_x is a free $\mathcal{O}_{\mathbb{P}^n,x}$-module by (11.3). □

□

Lemma 11.3 (Serre)**.** *Let A be a Noetherian local ring and let $I \lhd A$ be an ideal with free resolution of length* 1*:*

$$0 \longrightarrow A^p \longrightarrow A^q \longrightarrow I \longrightarrow 0.$$

If

$$0 \longrightarrow A \longrightarrow M \longrightarrow I \longrightarrow 0$$

represents $e \in \mathrm{Ext}^1_A(I, A)$ then M is locally free if and only if e generates the A-module $\mathrm{Ext}^1_A(I, A)$.

Proof. If we start with the short exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow I \longrightarrow 0$$

then we get a long exact sequence

$$\mathrm{Hom}_A(A, A) \longrightarrow \mathrm{Ext}^1_A(I, A) \longrightarrow \mathrm{Ext}^1_A(M, A) \longrightarrow \mathrm{Ext}^1_A(A, A) = 0.$$

Thus $\mathrm{Ext}^1_A(M, A) = 0$ if and only if the first map δ is surjective. Since $\delta(1) = e$, δ is surjective if and only if e generates the A-module $\mathrm{Ext}^1_A(I, A)$.

It remains to prove that if $\mathrm{Ext}^1_A(M, A) = 0$ then M is free. We have a pair of exact sequences

$$0 \longrightarrow A^p \longrightarrow A^q \longrightarrow I \longrightarrow 0.$$

and

$$0 \longrightarrow A \longrightarrow M \longrightarrow I \longrightarrow 0.$$

We lift the map $\phi\colon A^q \longrightarrow I$ to a map $\Phi\colon A^q \longrightarrow M$. Now define

$$\psi\colon (x, v) = \alpha(x) + \Phi(y),$$

where $\alpha\colon A \longrightarrow M$ is the first map. This gives us a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & A \oplus A^q & \longrightarrow & A^q & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle \Phi} & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & A & \xrightarrow[\alpha]{} & M & \xrightarrow[\beta]{} & I & \longrightarrow & 0
\end{array}$$

It follows that $\mathrm{Ker}\,\psi \simeq \mathrm{Ker}\,\phi \simeq A^p$ and $\mathrm{Coker}\,\psi = 0$. Thus we get an exact sequence

$$0 \longrightarrow A^p \longrightarrow A^r \longrightarrow M \longrightarrow 0$$

where $r = q + 1$. As $\mathrm{Ext}^1_A(M, A) = 0$, this sequence splits. Thus M is a direct summand of A^r, so that M is projective. As A is local it follows that M is free. $\square$