1. Rank one

We are going to switch freely between the holomorphic and algebraic perspective. This is justified by the following wonderful:

Theorem 1.1 (GAGA)**.** *Fix a projective variety X over* $\operatorname{Spec}\mathbb{C}$*.*

Then there is an equivalence of categories between the category of coherent sheaves on X and the category of coherent analytic sheaves on the underlying complex analytic space X^{an}.

Note that if the sheaves are the same then so is the cohomology. In particular global sections are the same. Even in the case of $\mathbb{P}^1$ and the structure sheaf this is quite striking. There are many more holomorphic functions on $\mathbb{C}$ than polynomial functions but every meromorphic function on $\mathbb{P}^1$ is given by a rational function.

Definition 1.2. *A **holomorphic vector bundle** E on a complex projective variety X is a complex manifold together with a holomorphic map $\pi\colon E \longrightarrow X$ and an open cover $\{ U_\alpha \}$ of X such that*

$$E|_{U_\alpha} = \pi^{-1}(U_\alpha)$$

is isomorphic to the product $U_\alpha \times \mathbb{C}^r$ over U_α,

$$\begin{array}{ccc} E|_{U_\alpha} & \longrightarrow & U_\alpha \times \mathbb{C}^r \\ & \searrow \quad \swarrow & \\ & U_\alpha, & \end{array}$$

such that on the overlap

$$U_{\alpha\beta} = U_\alpha \cap U_\beta,$$

the transition functions are linear functions on $\mathbb{C}^r$.

*The **rank** of E is r.*

In fact, vector bundles make sense in almost any geometric context. One can perform any operation on vector bundles, that makes sense for vector spaces. In particular, we can take the direct sum of two vector bundles, tensor product, Hom, dual, etc.

Given a holomorphic vector bundle E, we get a sheaf of sections,

Definition 1.3. *If E is a holomorphic vector bundle on a projective variety X then the associated **sheaf of sections** is the sheaf $\mathcal{O}_X(E)$ which assigns to the open subset $U \subset X$ the set of all holomorphic sections,*

$$\sigma\colon U \longrightarrow E$$

Note that the sheaf of sections is a locally free sheaf of rank r, the rank of E. Indeed, E is trivial over U then the sheaf of sections is a direct sum of r copies of $\mathcal{O}_U$.

There is an equivalence of categories between the category of holomorphic vector bundles on X and the category of locally free holomorphic sheaves on X. This equivalence respects the basic operations (direct sum, etc). The key point is that a vector bundle and a sheaf are both determined by a cover and the (same) transition functions.

By GAGA the holomorphic sheaf $\mathcal{O}_X(U)$ corresponds to a locally free algebraic sheaf (which, by abuse of notation, we will use the same symbol). Putting all this together, classifying holomorphic vector bundles on X is the same as classifying locally free coherent sheaves on X.

We recall the classification of line bundles on X, that is, rank one vector bundles. Suppose that L is a line bundle on X. By assumption we may find a cover $\{ U_\alpha \}$ of X such that $L_\alpha \simeq U_\alpha \times \mathbb{C}$. On overlaps we get a linear transformation of one dimensional vector spaces, that is, a nowhere zero holomorphic function, a one by one invertible matrix,

$$f_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathbb{C}^*.$$

On triple overlaps we have the following compatibility,

$$f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1.$$

By convention $f_{\alpha\alpha} = 1$ and $f_{\beta\alpha} = f_{\alpha\beta}^{-1}$.

In this way we get a 1-cocycle, with values in the sheaf of nowhere zero $\mathcal{O}_X^*$ holomorphic functions. Vice-versa, given a 1-cocycle $\sigma \in H^1(X, \mathcal{O}_X^*)$, by definition we are given an open cover $\{ U_\alpha \}$ of X and nohwere zero holomorphic functions

$$f_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathbb{C}^*.$$

subject to the rule

$$f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1.$$

Using this data, one can construct a holomorphic vector bundle with the given transition functions. (One can also use GAGA and construct the associated rank one locally free sheaf on the variety X).

On the other hand, one can take two line bundles and take the tensor product to get another line bundle. At the level of transition functions, one is just multiplying the transition functions. The trivial line bundle $X \times \mathbb{C}$, corresponding to the trivial sheaf $\mathcal{O}_X$, acts as the identity. The dual line bundle, $\operatorname{Hom}(L, X \times \mathbb{C})$ acts as the inverse; it is the line bundle with transition functions the reciprocal of the transition functions of L.

Definition 1.4. *Let X be a projective variety. The group of line bundles on X is called the* ***Picard group*** *and is denoted* $\operatorname{Pic}(X)$.

Theorem 1.5. *If X is a projective variety then*

$$\operatorname{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*).$$

One advantage of working over $\mathbb{C}$ is that there are more exact sequences. The exponential sequence is the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^*. \longrightarrow 0.$$

The map from the sheaf of holomorphic functions to the sheaf of nowhere holomorphic functions is the exponential,

$$f \longrightarrow \exp(2\pi i f).$$

The kernel is the locally constant sheaf $\mathbb{Z}$, the sheaf of integer valued holomorphic functions. As usual, a sequence of sheaves is exact if it is exact on stalks. Thus the exponential is surjective, as locally we can take logs.

If we take the long exact sequence of cohomology we get

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}).$$

Note that the sheaf cohomology groups

$$H^i(X, \mathbb{Z})$$

compute the usual topological cohomology. In particular these groups are finitely generated abelian groups. We have already observed that

$$H^1(X, \mathcal{O}_X^*) \simeq \operatorname{Pic}(X).$$

On the other hand,

$$H^1(X, \mathcal{O}_X)$$

is a finite dimensional vector space. The map

$$c_1 \colon \operatorname{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}),$$

is called the first chern class. It is a group homomorphism which assigns to every line bundle a cohomology class. If the line bundle L has a global section,

$$\sigma \in H^0(X, \mathcal{O}_X(L)),$$

we can assign the divisor D of zeroes of σ. Locally, just trivialise L and take the divisor of zeroes of the corresponding holomorphic function. On overlaps, the transition functions are nowhere zero holomorphic functions, so that even if we get different holomorphic functions, we get the same divisor of zeroes. In this case, the first chern class is the cocycle $[D]$ associated to the divisor D. In general, if H is an ample divisor then $L(kH)$ has global sections for k large enough. The first chern class of L is then the difference of $c_1(L(kH)$ and $[kH] = k[H]$, by linearity. Equivalently, every line bundle on a projective variety has a rational section, and the first chern class is the topological class of the divisor of zeroes minus poles of this rational section.

The kernel of the first chern class is the image of the vector space $H^1(X, \mathcal{O}_X)$. By Hodge theory, the free part of the abelian group $H^1(X, \mathbb{Z})$ is embedded as a lattice in $H^1(X, \mathcal{O}_X)$, and the quotient is an abelian variety, a projective algebraic group.

In fact, Grothendieck gave $\text{Pic}(X)$ the structure of a topological group. The quotient

$$\frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})}$$

is isomorphic to $\text{Pic}^0(X)$, the connected component of the identity. The quotient

$$\frac{\text{Pic}(X)}{\text{Pic}^0(X)}$$

is the group of connected components of $\text{Pic}(X)$ and it is embedded in $H^2(X, \mathbb{Z})$ by the first chern class.

To every Cartier divisor D, we can associate a line bundle $\mathcal{O}_X(D)$. In fact the data of a Cartier divisor gives rise a to 1-cocycle, which in turn gives rise to a line bundle. If $D \geq 0$ then $\mathcal{O}_X(D)$ comes with a section whose zero locus is precisely D. Two divisors D_1 and D_2 have isomorphic line bundles if and only if $D_1 \sim D_2$ are linearly equivalent. Thus the group of line bundles is isomorphic to the group of Cartier divisors modulo linear equivalence.

Theorem 1.6. $\text{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$.

Proof. The sheaf cohomology group

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0.$$

On the other hand,

$$H^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z},$$

generated by the class of a hyperplane. The sheaf locally free sheaf of rank one, $\mathcal{O}_{\mathbb{P}^n}(1)$ has a section with zero locus a hyperplane. Thus the first chern class map is an isomorphism. □