PRACTICE PROBLEMS FOR THE FIRST MIDTERM

1. Give the definition of:

(i) a divides b.
(ii) a prime number.
(iii) A Mersenne prime.
(iv) A group.
(v) A ring.
(vi) An integral domain.
(vii) The Fibonacci sequence.
(viii) $\tau(a)$.
(ix) $\sigma(a)$.
(x) The greatest common divisor.
(xi) Euclidean domain.
(xii) The least common multiple.

2. Show that if G is a set with a rule of multiplication which is associative and there is an element $e \in G$ such that $a \cdot e = a$, and there is an element b such that $a \cdot b = e$ for every $a \in G$ then G is a group.

3. Prove that the greatest common divisor of F_m and F_{m+1} is always one.

4. We say an integer is **square-free** if it is not divisible by the square of any prime.

Prove that every positive integer is uniquely the product of a square-free number and a square. Show that there are infinitely many square-free numbers.

5. Show that if $(b, c) = 1$ then

$$(a, bc) = (a, b)(a, c) \qquad \text{and} \qquad (bx + cy, bc) = (b, y)(c, x)$$

for all integers x and y.

6. Show that

$$(3 + \sqrt{10})^n$$

is a unit in $\mathbb{Z}[\sqrt{10}]$ for every $n \in \mathbb{Z}$.

7. Show that if a, b and c are natural numbers and $(a, b) = 1$ then the number n of non-negative solutions of

$$ax + by = c,$$

satisfies the inequality

$$\frac{c}{ab} - 1 < n \leq \frac{c}{ab} + 1.$$