MODEL ANSWERS TO THE FIFTH HOMEWORK

1. If $M \simeq F \oplus T$ is the decomposition of M into its free and torsion part then $F \simeq R^r$ and the rank r is the dimension of the K-vector space $M \underset{R}{\otimes} K$. Let s be the rank of N. The rank of $M \oplus N$ is

$$\begin{aligned}\dim_K((M \oplus N) \underset{R}{\otimes} K) &= \dim_K(M \underset{R}{\otimes} K \oplus N \underset{R}{\otimes} K)\\ &= \dim_K(M \underset{R}{\otimes} K) + \dim_K(N \underset{R}{\otimes} K)\\ &= r + s.\end{aligned}$$

2. A rational canonical form determines a partition of 5, $p_1, p_2, \dots, p_k$. Since d_i divides d_{i+1} we have that

$$p_1 \leq p_2 \leq p_3 \dots \qquad \text{and} \qquad \sum p_i = 5.$$

The possible partition types are

1+1+1+1+1; 1+1+1+2; 1+1+3; 1+2+2; 1+4; 2+3; 5.

For each one we count the number of different rational canonical forms with the given partition type.
In the first case, $d_1 = d_2 = d_3 = d_4 = d_5$ has degree one; $d_1 = x$ or $x + 1$ and we either have the zero matrix or the identity matrix.
In the second case $d_1 = d_2 = d_3$ has degree one and d_4 has degree two. There are two possibilities for d_1 and two possibilities for d_4/d_1. Thus there are four different matrices.
In the third case $d_1 = d_2$ has degree one and d_3/d_1 has degree two. There are two possibilities for d_1 and four for d_3/d_1. Thus there are eight different matrices.
In the fourth case d_1 has degree one, d_2/d_1 has degree one and $d_2 = d_3$. There are 2 possibilities for d_1 and two for d_2/d_1. There are four different matrices.
In the fifth case d_1 has degree one and d_2/d_1 has degree three. There are 2 possibilities for d_1 and eight for d_2/d_1. There are sixteen different matrices.
In the sixth case d_1 has degree 2 and d_2/d_1 has degree one. There are four possibilities for d_2 and two for d_2/d_1. There are eight different matrices.
In this seventh case d_1 has degree five. There are 32 such matrices.

In total there are

$$2+4+8+4+16+8+32=74$$

rational canonical forms.

3. The matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is in rational canonical form. Both the minimal and characteristic polynomials are therefore $x^4 - 1$, up to sign, so that the eigenvalues are ± 1 and $\pm i$.

4. The vector $(0, 1, -1)$ is an eigenvector with eigenvalue zero, and the vector $(1, 0, 1)$ is an eigenvector with eigenvalue -1. The sum of the eigenvalues is the trace, which is 0, so that there must be an eigenvector with eigenvalue 1. It follows that the minimum polynomial is $x(x-1)(x+1) = x^3 - x$.

The rational canonical form is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

5. Find the Jordan canonical form of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}.$$

As this matrix has block form with a 1×1 matrix at the top left, it obviously suffices to put the 2×2 submatrix

$$\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix},$$

into Jordan canonical form. The last matrix has characteristic polynomial

$$\begin{vmatrix} -x & -2 \\ 1 & 3-x \end{vmatrix} = x(x-3) + 2 = x^2 - 3x + 2 = (x-2)(x-1).$$

This has eigenvalues 1 and 2. It follows that the original matrix has eigenvalues 1, 1 and 2. As there are two independent eigenvectors with eigenvalue 1, it follows that the Jordan canonical form is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$