MODEL ANSWERS TO THE FOURTH HOMEWORK

1. (i) We first try to define a map left to right. Since finite direct sums are the same as direct products, we just have to define a map to each factor. By the universal property of the wedge product we just have to define a map from the Cartesian product of $M \oplus N$ with itself d times which is multilinear and alternating:

$$(M \oplus N)^d \longrightarrow \bigwedge^i M \underset{R}{\otimes} \bigwedge^j N.$$

To define this map, we work by analogy with the binomial theorem. If we expand $(a+b)^d$ using the distributive rule, then we terms of the form $a^i b^j$ by taking a from i factors and b from the rest. The one difference for the wedge product is we need to keep track of sign. We send a d-tuple

$$(m_i, n_i)_{i=1}^d$$

to

$$\sum_{I,J} \epsilon_{I,J} m_I \otimes n_J.$$

The sum runs over all partitions (I, J) of the integers into two pieces of sizes i and j. $\epsilon_{I,J}$ is the sign of the permutation given by I and J (in other words, imagine arranging a deck of cards, first put down the elements of I and then the elements of J, in increasing order). m_I denotes the wedge product of m_i, as i runs over the elements of I.
It is not hard to check that this is multilinear and alternating.
To define a map right to left, we proceed in a similar fashion. It suffices to define a bilinear map

$$\bigwedge^i M \times \bigwedge^j N \longrightarrow \bigwedge^d (M \oplus N).$$

We send (m_I, n_J) to $\epsilon_{I,J}(m_{a_1}, 0) \wedge (m_{a_2}, 0) \wedge \cdots \wedge (0, n_{b_j})$.
To finish it suffices to check that the composition from

$$\bigwedge^i M \underset{R}{\otimes} \bigwedge^j N \longrightarrow \bigwedge^i M \underset{R}{\otimes} \bigwedge^j N,$$

is $\pm$ times the identity.

(ii) By the universal property of the tensor product, an element of $\mathrm{Hom}_R(M \underset{R}{\otimes} N, P)$ is the same as a bilinear map

$$M \times N \longrightarrow P.$$

If we fix $m \in M$ this gives us an R-linear map $N \longrightarrow P$, an element of $\mathrm{Hom}_R(N, P)$. Varying m gives us a function

$$M \longrightarrow \mathrm{Hom}_R(N, P),$$

which it is not hard to see is R-linear. Thus we get an element of $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. It is straightforward to check that this assignment is R-linear.
Now suppose that we have an element of $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. For every $m \in M$ we get an R-linear map $N \longrightarrow P$. This defines a function $M \times N \longrightarrow P$ which is bilinear, so that we get an element of $\mathrm{Hom}_R(M \underset{R}{\otimes} N, P)$. It is not hard to see that this is the inverse of the first assignment, so that we get an isomorphism:

$$\mathrm{Hom}_R(M \underset{R}{\otimes} N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

2. We first construct a map. By the universal property of the tensor product, we just need to exhibit a bilinear map

$$L\colon V^* \times W \longrightarrow \mathrm{Hom}_F(V, W).$$

Given a pair (ϕ, w) send this to the linear function $\psi(v) = \phi(v)w$. It is not hard to check that this map is bilinear.
Suppose we are given an element of $V^* \underset{F}{\otimes} W$.

$$\sum \phi_i \otimes w_i,$$

where $\phi_i \in V^*$ is a linear functional on V and $w_i \in W$. We send this to the linear map

$$\psi\colon V \longrightarrow W \qquad \text{given by} \qquad \psi(v) = \sum \phi_i(v) w_i.$$

The image of ψ is contained in the span of the vectors $w_1, w_2, \dots, w_k$ so that $\psi \in \mathrm{Hom}^f_F(V, W)$.
Suppose that W_0 and W_1 are complimentary linear subspaces. Then

$$V^* \underset{F}{\otimes} W \simeq V^* \underset{F}{\otimes} W_0 \oplus V^* \underset{F}{\otimes} W_1$$

and

$$\mathrm{Hom}_F(V, W) \simeq \mathrm{Hom}_F(V, W_0) \oplus \mathrm{Hom}_F(V, W_1).$$

Moreover L respects this decomposition.

Now we check that L is injective and surjective. Suppose first that W is finite dimensional. In this case $W \simeq F^r$, for some r. As

$$\begin{aligned} V^* \underset{F}{\otimes} W &\simeq V^* \underset{F}{\otimes} F^r \\ &\simeq (V^* \underset{F}{\otimes} F)^r \\ &\simeq (V^*)^r, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Hom}^f_F(V, W) &= \operatorname{Hom}_F(V, W) \\ &\simeq \operatorname{Hom}_F(V, F^r) \\ &\simeq (\operatorname{Hom}_F(V, F))^r \\ &\simeq (V^*)^r, \end{aligned}$$

it is clear that L is an isomorphism in this case.
Now consider the general case. If $\psi \in \operatorname{Hom}^f_F(V, W)$ then we want to construct an element of $V^* \underset{F}{\otimes} W$ mapping to ψ. Let W_0 be the image of ψ and let $\psi_0 \colon V \longrightarrow W_0$ be the obvious factorisation of $V \longrightarrow W$. Then we may find $\alpha_0 \in V^* \underset{F}{\otimes} W_0$ such that α_0 is sent to ψ_0. Let α be the image of α_0 in $V^* \underset{F}{\otimes} W$. It is clear that $L(\alpha) = \psi$. Thus L is surjective.
Now suppose that we have an element $\alpha \in V^* \underset{F}{\otimes} W$ of the kernel of L. Then

$$\alpha = \sum \phi_i \otimes w_i.$$

Let W_0 be the span of $w_1, w_2, \dots, w_k$. Then we may find $\alpha_0 \in V^* \underset{F}{\otimes} W_0$ mapping to α such that the image of α_0 in $\operatorname{Hom}_F(V, W_0)$ is zero. As W_0 is finite dimensional, it follows that α_0 is zero.
As W is a vector space, we may find W_1 a complimentary linear subspace such that $W = W_0 + W_1$. As L respects this decomposition, it follows that α is zero so that it must be zero to begin with.
3. First suppose that

$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

is right exact. Call the first map $f \colon M \longrightarrow N$ and the second map $g \colon N \longrightarrow P$. Note that $\operatorname{Im} f \subset \operatorname{Ker} g$ is equivalent to $g \circ f = 0$.
Let Q be any R-module. Suppose that $\alpha \in \operatorname{Hom}_R(P, Q)$. Then the image of α is $\beta = g^*(\alpha) = \alpha \circ g$. If α is non-zero then we may find $p \in P$ such that $\alpha(p) \neq 0$. As g is surjective we may find $n \in N$ such that $g(n) = p$. In this case $\beta(n) = \alpha(g(n)) = \alpha(p) \neq 0$. Thus $\beta \neq 0$ and so we have exactness on the left.

Now consider the image γ of α in $\operatorname{Hom}_R(M, Q)$. $\gamma = \alpha \circ (g \circ f)$. But $g \circ f = 0$ by exactness, so that $\gamma = 0$. Finally suppose that $\beta \in \operatorname{Hom}_R(N, Q)$ and the image of β is zero, so that $\beta \circ f = 0$. Then β is zero on the image of f, so that β is zero on the kernel of g. By the universal property of the quotient, it follows that β induces a morphism $\alpha \colon M \longrightarrow Q$ such that $\beta = \alpha \circ g$. Thus β is the image of α and so we have a left exact sequence.
Now suppose that

$$0 \longrightarrow \operatorname{Hom}_R(P, Q) \longrightarrow \operatorname{Hom}_R(N, Q) \longrightarrow \operatorname{Hom}_R(M, Q),$$

is left exact for every R-module Q.
Suppose we take $Q = M$ and $\gamma \in \operatorname{Hom}_R(M, M)$ to be the identity. Then we can find $\beta \in \operatorname{Hom}_R(N, Q)$ such that $\gamma = \beta \circ f$. As γ is injective, it follows that f is injective.
Now take $Q = P$ and let α be the identity. Then $g \circ f \circ \alpha = 0$ so that $g \circ f = 0$. Finally take $Q = N/\operatorname{Im} f$ and let β be the canonical map $N \longrightarrow Q$. Then the image of β in $\operatorname{Hom}_R(M, Q)$ is zero. So we may find $\alpha P \longrightarrow Q$ mapping to β, so that $\beta = g \circ \alpha$. This is only possible if the image of f contains the kernel of g. Thus the first seqeunce is exact.
4. Suppose that the first sequence is left exact. Let Q be R-module. If $\alpha \in \operatorname{Hom}_R(Q, M)$ is non-zero then we may find $q \in Q$ such that $\alpha(q) \neq 0$. If $\beta = f \circ \alpha$ then $\beta(q) = f(\alpha(q)) \neq 0$ as f is injective.
If γ is the image of α in $\operatorname{Hom}_R(Q, P)$ then

$$\gamma = g \circ (f \circ \alpha) = (g \circ f) \circ \alpha = 0,$$

as $g \circ f = 0$. If $\beta \in \operatorname{Hom}_R(Q, N)$ and $g \circ \beta = 0$ then the image of β lands in the kernel of g. As this is equal to the image of f, the image of β must land in the image of f. As f defines an isomorphism with its image, we can find $\alpha \in \operatorname{Hom}_R(Q, M)$ such that $f \circ \alpha = \beta$. Thus the second sequence is always exact.
Now suppose that the second sequence is always left exact. If $Q = R$ then

$$0 \longrightarrow \operatorname{Hom}_R(R, M) \longrightarrow \operatorname{Hom}_R(R, N) \longrightarrow \operatorname{Hom}_R(R, P),$$

is exact. On the other hand, the map

$$M \longrightarrow \operatorname{Hom}_R(R, M),$$

which sends m to the R-linear map $r \longrightarrow rm$ is an isomorphism. Thus we recover the first exact sequence, which is then automatically left exact.

5. Suppose that the first sequence is right exact. If Q is an R-module and Q_0 is any other R-module then let

$$Q_1 = \operatorname{Hom}_R(Q, Q_0).$$

It follows by (3) that the sequence

$$0 \longrightarrow \operatorname{Hom}_R(P, Q_1) \longrightarrow \operatorname{Hom}_R(N, Q_1) \longrightarrow \operatorname{Hom}_R(M, Q_1),$$

is always left exact. By (1) (ii) it follows that

$$0 \longrightarrow \operatorname{Hom}_R(P \underset{R}{\otimes} Q, Q_0) \longrightarrow \operatorname{Hom}_R(N \underset{R}{\otimes} Q, Q_0) \longrightarrow \operatorname{Hom}_R(M \underset{R}{\otimes} Q, Q_0),$$

is always left exact. By (3) it follows that

$$M \underset{R}{\otimes} Q \longrightarrow N \underset{R}{\otimes} Q \longrightarrow P \underset{R}{\otimes} Q \longrightarrow 0.$$

is right exact for all R-modules Q.
Now suppose that the first sequence is right exact. If we take $Q = R$ then we recover the first exact sequence, since tensoring with R has no effect.
6. The key exact sequence is the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

of $\mathbb{Z}$-modules, where the first map is multiplication by 2. We take $Q = \mathbb{Z}_2$. First we look at maps into Q. Since

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_2) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2,$$

we get the sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

This is surely not a short exact sequence.
Now suppose we take $Q = \mathbb{Z}_2$ and look at maps from Q. Since

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) = 0,$$

we get the sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

This is surely not a short exact sequence.
Finally suppose we tensor by $Q = \mathbb{Z}_2$. Then we get the first sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

which is still not a short exact sequence.
Challenge Problem: 7. The key point is to write down a PID which is not a Euclidean domain. The ring of integers of $\sqrt{-19}$ has this

property. It consists of all integral linear combinations of $1/2$ and $\sqrt{-19}/2$. If we put

$$\theta = \frac{1+\sqrt{-19}}{2}$$

then this is the same as the ring $\mathbb{Z}[\theta]$.

It is easy to check that the units in this ring are ± 1. It is a non-trivial fact that this ring is a PID.

Consider the matrix

$$A = \begin{pmatrix} \theta \\ 2 \end{pmatrix}.$$

Note that $\theta\bar{\theta} = 5$. Thus

$$(\bar{\theta}) \cdot \theta + (-2) \cdot 2 = 1,$$

and the gcd of θ and 2 is 1. Let

$$B = \begin{pmatrix} \bar{\theta} & -2 \\ -2 & \theta \end{pmatrix}.$$

Then $\det B = 1$ so that B is invertible and

$$BA = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We claim that one cannot row reduce A to this form. Suppose that

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an invertible matrix such that

$$CA = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As 2 and θ are coprime, (c, d) must be a multiple of $(-2, \theta)$. As C is invertible, the determinant of C is a unit. It follows that (c, d) are coprime. Thus $(c, d) = \pm(-2, \theta)$. Thus the top row must be of the form

$$(a, b) = \pm(\bar{\theta}, -2) + q(\theta, 2),$$

where q is an arbitrary element of the ring.

If we can row reduce A to e_1 then we can choose C so that it is product of lower and upper triangular matrices, a permutation matrix and an invertible diagonal matrix.