MODEL ANSWERS TO THE THIRD HOMEWORK

1. (a) We have to define an R-linear map,

$$\phi\colon M \underset{R}{\otimes} N \longrightarrow N \underset{R}{\otimes} M.$$

By the universal property of $M \underset{R}{\otimes} N$, it suffices to exhibit a bilinear map

$$f\colon M \times N \longrightarrow N \underset{R}{\otimes} M$$

The composition of $u\colon N \times M \longrightarrow N \underset{R}{\otimes} M$ and the map

$$M \times N \longrightarrow N \times M \qquad \text{which sends} \qquad (m,n) \longrightarrow (n,m)$$

will obviously do. The inverse map is constructed similarly. The composition either way is easily seen to be the identity, either because it satisfies the universal property of the identity, or because it is the identity map on generators.
(b) One can prove this as above. Here is a much sneakier way to proceed. Note the canonical isomorphism,

$$(M \times N) \times P \simeq M \times (N \times P).$$

On the other hand, given either triple product, one can consider trilinear maps, that is maps that are linear in all three variables. It is not hard to check that $(M \underset{R}{\otimes} N) \underset{R}{\otimes} P$ satisfies the corresponding universal propery. Similarly for $M \underset{R}{\otimes} (N \underset{R}{\otimes} P)$. Thus they are canonically isomorphic.
(c) We are going to show that M satisfies the properties of the tensor product. First we need to exhibit a bilinear map,

$$u\colon R \times M \longrightarrow M$$

The definition of u is almost forced, send (r,m) to rm. This is clearly a bilinear map. Now suppose we are given a bilinear map

$$f\colon R \times M \longrightarrow N.$$

Define

$$\phi\colon M \longrightarrow N$$

by sending m to $f(1,m)$. We check that the diagram,

$$\begin{tikzcd} R \times M \arrow[r, "f"] \arrow[d, "u"'] & N \\ M \arrow[ur, "\phi"] & \end{tikzcd}$$

commutes. Suppose that $(r,m) \in R \times M$. Then

$$\begin{aligned} \phi \circ u(r,m) &= \phi(rm) \\ &= f(1,rm) \\ &= rf(1,m) \\ &= f(r,m), \end{aligned}$$

where we applied bilinearity of f twice. Thus the diagram commutes. Finally we check that ϕ is R-linear. Suppose that m_1, $m_2 \in M$. Then

$$\begin{aligned} \phi(m_1+m_2) &= f(1,m_1+m_2) \\ &= f(1,m_1)+f(1,m_2) \\ &= \phi(m_1)+\phi(m_2). \end{aligned}$$

Now suppose that $r \in R$ and $m \in M$. Then

$$\begin{aligned} \phi(rm) &= f(1,rm) \\ &= rf(1,m) \\ &= r\phi(m). \end{aligned}$$

Thus ϕ is R-linear. Thus M satisfies all the properties of a tensor product and the result is clear.
(d) First we define a bilinear map

$$M \times (N \oplus P) \longrightarrow (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P),$$

by sending $(m,(n,p))$ to $(m \otimes n, m \otimes p)$. It is easy to check that this is bilinear. This gives us a map one way. To get a map the other way, it suffices, by definition of the direct sum and then of the tensor product and by symmetry, to exhibit a bilinear map

$$M \times N \longrightarrow M \underset{R}{\otimes} (N \oplus P).$$

For this send (m,n) to $m \otimes (n,0)$. Again it is clear that this map is bilinear and that the induced R-linear maps are inverse to each other.
(e) As $F \simeq R^n$, this follows immediately from (c) and (d), by induction on n.

2. Let d be the gcd of m and n. I claim that

$$\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_m \simeq \mathbb{Z}_d.$$

The proof proceeds in two steps. First observe that

$$\begin{aligned} m(1 \otimes 1) &= m \otimes 1 \\ &= 0 \otimes 1 \\ &= 0. \end{aligned}$$

Similarly $n(1 \otimes 1) = 0$. As $\mathbb{Z}$ is a PID, we may find r and s such that

$$d = rm + sn.$$

Thus

$$\begin{aligned} d(1 \otimes 1) &= (rm + sn)1 \otimes 1 \\ &= r(m(1 \otimes 1) + s(n(1 \otimes 1)) \\ &= 0. \end{aligned}$$

Thus $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_m$ is surely isomorphic to a subgroup of $\mathbb{Z}_d$. It remains to check that no smaller multiple of $1 \otimes 1$ is zero. The best way to prove this is to use the universal property. Let

$$f \colon \mathbb{Z}_m \times \mathbb{Z}_m \longrightarrow \mathbb{Z}_d$$

be the map that sends (a, b) to ab. As d divides both m and n, this map is indeed well-defined. On the other it is clearly bilinear. By the universal property, it induces an R-linear map

$$\phi \colon \mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_m \longrightarrow \mathbb{Z}_d.$$

This map sends $1 \otimes 1$ to $f(1, 1)$, that is, 1. Hence if $k(1 \otimes 1) = 0$, then k is zero in $\mathbb{Z}_d$ and so d divides k. The result follows.

3. We first prove that $M \underset{R}{\otimes} N$ is finitely generated. Suppose that $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_n$ are generators of M and N. Then I claim that $x_i \otimes y_j$ are generators of $M \underset{R}{\otimes} N$. Indeed this is generated by elements of the form $m \otimes n$, and so it is enough to observe that if

$$m = \sum r_i x_i \qquad \text{and} \qquad n = \sum s_i n_i,$$

then

$$m \otimes n = \sum r_i s_j x_i \otimes y_j,$$

where of course we use bilinearity to distribute the sum.

As R is Noetherian, $M \underset{R}{\otimes} N$ is Noetherian, as it is a finitely generated module over a Noetherian ring.

But in fact we don't need the fact that R is Noetherian. First observe that the direct sum M^n is Noetherian, by induction on n, applied to the standard short exact sequence:

$$0 \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M \longrightarrow 0.$$

It follows that

$$M \underset{R}{\otimes} R^n \simeq (M \underset{R}{\otimes} R)^n \simeq M^n.,$$

is Noetherian. By assumption there is a surjective R-linear map

$$R^n \longrightarrow N,$$

for some n. If we tensor this by M we get a surjective R-linear map (it is a general fact that surjective maps remain surjective after tensoring, see the next hwk)

$$M \underset{R}{\otimes} R^n \longrightarrow M \underset{R}{\otimes} N.$$

Thus $M \underset{R}{\otimes} N$ is a quotient of a Noetherian R-module, so that it is Noetherian.

4. An $\mathbb{R}$-module is the same as a real vector space. Vector spaces are classified by their dimension.

$\mathbb{C}$ is a two dimensional vector space over $\mathbb{R}$. Therefore

$$\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$$

is a four dimensional real vector space.

$$\mathbb{C} \underset{\mathbb{C}}{\otimes} \mathbb{C}$$

is a one dimensional complex vector space. Considered as a real vector space it is two dimensional.

Thus $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$ and $\mathbb{C} \underset{\mathbb{C}}{\otimes} \mathbb{C}$ are not isomorphic $\mathbb{R}$-modules.

5. Once again a module over a field is nothing more than a vector space, so we just have to show these have the same dimension.

Consider

$$\frac{a}{b} \otimes \frac{c}{d} \in \mathbb{Q} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

where a, b, c and d are all integers, and $bd \neq 0$. We have

$$\frac{a}{b} \otimes \frac{c}{d} = \frac{c}{d}\frac{a}{b} \otimes 1.$$

Thus every element of $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is equivalent to

$$\frac{a}{b} \otimes 1.$$

It follows that $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a one dimensional vector space over $\mathbb{Q}$. On the other hand, $\mathbb{Q} \underset{\mathbb{Q}}{\otimes} \mathbb{Q}$ is also a one dimensional vector space over $\mathbb{Q}$,

6. Consider

$$\frac{a}{b} \otimes \frac{c}{d},$$

where a, b, c and d are all integers, and $bd \neq 0$. We may suppose that $d > 0$. In this case

$$\begin{aligned}
0 &= \frac{a}{bd} \otimes 0 \\
&= \frac{a}{bd} \otimes c \\
&= d(\frac{a}{bd} \otimes \frac{c}{d}) \\
&= \frac{a}{b} \otimes \frac{c}{d}.
\end{aligned}$$

Thus

$$\mathbb{Q}/\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} \simeq 0.$$

Challenge Problem: 7. Let M_i be a collection of R-modules, indexed by a set I. Let M be any R-module. We claim that there is a natural isomorphism

$$\bigoplus_{i \in I} (M_i \underset{R}{\otimes} M) \simeq (\bigoplus_{i \in I} M_i) \underset{R}{\otimes} M.$$

To define an R-linear map left to right, by the universal property of the direct sum, we just need to define an R-linear map

$$M_i \underset{R}{\otimes} M \longrightarrow (\bigoplus_{i \in I} M_i) \underset{R}{\otimes} M,$$

for each index i. By the universal property of the tensor product it suffices to define a bilinear map

$$M_i \times M \longrightarrow (\bigoplus_{i \in I} M_i) \underset{R}{\otimes} M,$$

We just send (m_i, m) to the element $m_i' \otimes m$, where m_i' is the element of $\bigoplus_{i \in I} M_i$ with the entry $m_i \in M_i$ and 0 everywhere else. This map is clearly bilinear.

To define a map right to left we just need to define a bilinear map

$$(\bigoplus_{i \in I} M_i) \times M \longrightarrow \bigoplus_{i \in I} (M_i \underset{R}{\otimes} M).$$

We just send $((m_i)_{i \in I}, m)$ to $(m_i \otimes m)_{i \in I}$. By definition of the direct sum, $m_i = 0$ for all but finitely many i, so that $m_i \otimes m = 0$ for all but finitely many i.

Thus we get R-linear maps in both directions, which are easily seen to be inverse maps.
If we replace the direct sum by the product, then we can still define an R-linear map right to left but it is not so clear how to go left to right.
Let $R = \mathbb{Z}$, $M_i = \mathbb{Z}_i$ and $M = \mathbb{Q}$. Note that

$$M_i \underset{R}{\otimes} M = 0.$$

Indeed the LHS is a vector space over $\mathbb{Q}$, by extension of scalars. It is also a torsion abelian group; every element has order dividing i. The only vector space over $\mathbb{Q}$ where every element is torsion is the trivial vector space.
Thus

$$\prod_{i \in I} (M_i \underset{R}{\otimes} M) = 0.$$

On the other hand,

$$\prod_{i \in I} M_i$$

has elements of infinite order. The order $(m_i)_{i \in I}$, where $m_i = 1$ for all i, has infinite order. This gives us an injective $\mathbb{Z}$-linear map

$$\mathbb{Z} \longrightarrow \prod_{i \in I} M_i$$

The key point is that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, meaning that when we tensor this injective map by $\mathbb{Q}$, it remains injective. Therefore, if we tensor this with $\mathbb{Q}$, we get a vector space of positive dimension.