MODEL ANSWERS TO THE FIRST HOMEWORK

1. Define a function

$$d\colon F[[x]] - \{0\} \longrightarrow \mathbb{N} \cup \{0\}$$

by sending a power series to its degree. We have to check two things. It is easy to see that if f and g are non-zero power series then $d(f) \leq d(fg)$.
Otherwise we have to check that if $f(x)$ and $g(x)$ are two power series, then we may find $q(x)$ and $r(x)$ such that

$$g(x) = q(x)f(x) + r(x),$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $f(x)$. There are two cases. If the degree of $g(x)$ is less than the degree of $f(x)$ there is nothing to do; take $q(x) = 0$ and $r(x) = g(x)$. In this case the fact that $r(x)$ has degree less than $f(x)$ is clear.
Otherwise I claim that $f(x)$ divides perfectly into $g(x)$. To see this, note that we have

$$\begin{aligned} f(x) &= ax^d + \dots \\ &= x^d(a + \dots) \\ &= x^d u. \end{aligned}$$

Here as $a \neq 0$, and F is a field, a is a unit. Thus u is a unit. But then by the same token, $g(x) = x^e v$, where e is the degree of g and v is a unit. Thus

$$g(x) = q(x)f(x),$$

where $q(x) = x^{e-d}vw$ and w is the inverse of u. Thus we have a Euclidean Domain.
2. Suppose that M is an R-module. Then M is a vector space over F, call it V. Multiplication by x induces a linear map

$$\phi\colon V \longrightarrow V.$$

Now suppose that we are given a linear map

$$\phi\colon V \longrightarrow V.$$

Given $f(x) \in R$, we need to define a multiplication map,

$$m\colon V \longrightarrow V.$$

We send $v \in V$ to
$$f(\phi)(v).$$
3. Let F be the set of all functions from X to M. We need to define a rule of addition and scalar multiplication. Suppose that f and g are elements of M. Define $f+g$ as the pointwise sum, so that
$$(f+g)(x) = f(x) + g(x).$$
Similarly, given $r \in R$ and $f \in F$, define rf as the pointwise product,
$$(rf)(x) = r(f(x)).$$
It is an easy matter to check that with this rule of addition and scalar multiplication, F becomes an R-module.

Let $H = \mathrm{Hom}_R(M, N)$ be the set of all R-module homomorphisms. Then H is a subset of F, the set of all functions from M to N. It suffices to prove that H is non-empty and closed under addition and scalar multiplication.

First note that the zero map, which sends every element of M to the zero element of N, is R-linear. Thus H is certainly non-empty. Suppose that f and g are elements of H. We need to prove that $f+g$ is R-linear. Let m and n be elements of M and r and s be elements of R. We have
$$\begin{aligned}(f+g)(rm+sn) &= f(rm+sn) + g(rm+sn)\\ &= rf(m) + sf(n) + rg(m) + sg(n)\\ &= rf(m) + rg(m) + sf(m) + sf(n)\\ &= r(f+g)(m) + s(f+g)(n).\end{aligned}$$
Thus $f+g$ is indeed R-linear. It is equally easy and just as formal to prove that rf is R-linear. Thus H is closed under addition and scalar multiplication and so H is an R-module.

4. Since the arbitrary intersection of ideals is an ideal, it suffices to prove that I is an ideal, in the case that X contains one point x. Clearly $0 \in I$. Thus I is non-empty. Suppose that i and j are elements of I. Then
$$\begin{aligned}(i+j)(x) &= ix + jx\\ &= 0 + 0 = 0.\end{aligned}$$
Thus $i + j \in I$ and I is closed under additition. Now suppose that $r \in R$ and $i \in I$. Then
$$\begin{aligned}ri(x) &= r(ix)\\ &= r0\\ &= 0.\end{aligned}$$

Thus $ri \in I$ and I is an ideal. Here is another way to conclude that I is an ideal. Let

$$\phi\colon R \longrightarrow \operatorname{Hom}_R(M, M)$$

be the natural map which sends an element R to the R-linear map, $m \longrightarrow rm$. It is easy to see that ϕ is R-linear. Replacing M by the module generated by X, note that an element $r \in R$ is in I if and only if $\phi(r)$ is the zero map. Thus I is the kernel of ϕ. It also follows that I is also the annihilator of $\langle X \rangle$.