4. Symmetric and Alternating Products

We want to introduce some variations on the theme of tensor products.

The idea is that one can require a bilinear map to be either symmetric or alternating.

Definition 4.1. *Let M and P be two R-modules and let*

$$f\colon M \times M \longrightarrow P$$

*be a bilinear map. We say that f is **symmetric** if*

$$f(m,n) = f(n,m).$$

*We say that f is **alternating** if*

$$f(m,n) = -f(n,m).$$

Definition 4.2. *Let M be an R-module. The **symmetric product** of M with itself, denoted $\operatorname{Sym}^2 M$, is an R-module, together with a symmetric bilinear map*

$$u\colon M \times M \longrightarrow \operatorname{Sym}^2 M,$$

which is universal amongst all symmetric bilinear maps, in the following sense: let

$$f\colon M \times M \longrightarrow P,$$

be any other symmetric bilinear map. Then there is a unique induced R-linear map

$$\phi\colon \operatorname{Sym}^2 M \longrightarrow P,$$

which makes the following diagram commute

$$\begin{array}{ccc} M \times M & \xrightarrow{f} & P. \\ {\scriptstyle u}\downarrow & \nearrow{\scriptstyle \phi} & \\ \operatorname{Sym}^2 M. & & \end{array}$$

For the usual reasons, the symmetric product is unique, up to unique isomorphism, if it exists at all. Note also that there is an R-linear map

$$M \underset{R}{\otimes} M \longrightarrow \operatorname{Sym}^2 M,$$

whose existence is guaranteed by the universal property of the tensor product, once again given that the symmetric product exists at all. This suggests that the construction of the symmetric product goes along the same lines as the tensor product, except that one introduces more relations.

Lemma 4.3. *Let M be an R-module.*

Then the symmetric product exists.

Proof. Let F be the free module with generators every element of $M \times M$, and let G' be the submodule generated by G (that is all the old relations) union the extra relations

$$(m, n) - (n, m).$$

Define the symmetric product to be the quotient F/G'. It is left as an exercise to the reader to check that this is indeed the symmetric product. □

Definition 4.4. *Let M be an R-module. The* ***wedge product*** *of M with itself, denoted $\bigwedge^2 M$, is an R-module, together with a skew-symmetric map*

$$u\colon M \times M \longrightarrow \bigwedge^2 M$$

which is universal amongst all such skew-symmetric bilinear map in the following sense: given any skew-symmetric bilinear map

$$f\colon M \times M \longrightarrow P$$

there is a unique R-linear map

$$\phi\colon \bigwedge^2 M \longrightarrow P,$$

which makes the standard diagram commute.

Uniqueness follows by the standard arguments; existence parrallels the construction of the symmetric product, the only difference being that we throw in the generators

$$(m, n) + (n, m)$$

instead of

$$(m, n) - (n, m).$$

In both cases, it is customary to employ notation for the image of (m, n). In the case of the symmetric product, we have

$$m \cdot n$$

which is subject to the rule

$$m \cdot n = n \cdot m.$$

In the case of the wedge product,

$$m \wedge n = -m \wedge n.$$

Note that if 2 is invertible in R, then

$$m \wedge m = 0.$$

Perhaps one of the most interesting uses of the symmetric and alternating product, is in the case of vector spaces. If V is a vector space over a field, not of characteristic two, and $e_1, e_2, \dots, e_n$ is a basis for V, then

$$e_i \cdot e_j$$

is a basis for $\operatorname{Sym}^2 V$, where $1 \leq i \leq j \leq n$ and

$$e_i \wedge e_j$$

is a basis for $\bigwedge^2 V$, where $1 \leq i < j \leq n$. In particular $\wedge^2 V$ has dimension

$$\binom{n}{2}.$$

Definition 4.5. *Let*

$$f \colon M_1 \times M_2 \times \dots \times M_d \longrightarrow N$$

*be a map. We say that f is **multilinear** if it is linear in each variable.*

*If $M_1 = M_2 = \dots = M_d$, and f is invariant (respectively changes sign) whenever two coordinates are switched, then we say that f is **symmetric** (respectively **alternating**).*

There are correspondingly three associated universal objects. The first is in fact isomorphic to the tensor product of $M_1, M_2, \dots, M_d$ (note that since the tensor product is an associative operation, in fact it makes sense to talk about the d-fold product, without specifying an order). The second and third are $\operatorname{Sym}^d M$ and $\bigwedge^d M$.

Definition 4.6. *Let M be an R-module and let $\phi \colon M \longrightarrow N$ be an R-linear map. Let f be the composition of the natural map*

$$\phi^n \colon M \times M \times \dots \times M \longrightarrow N \times N \times \dots \times N$$

and

$$u \colon N \times N \times \dots \times N \longrightarrow \bigwedge^n N.$$

Then f is alternating bilinear. By the universal property of $\bigwedge^n M$, there is an induced R-linear map,

$$\bigwedge^n \phi \colon \bigwedge^n M \longrightarrow \bigwedge^n N.$$

Put differently, there is a covariant functor F from the category of R-modules to itself, which associates to any module M, the module $\bigwedge^n M$.

Note one interesting thing about the construction of $\wedge^n \phi$. Suppose that we go back to the case of a vector space V. If V has dimension n, then in fact $\wedge^i V$ has dimension

$$\binom{n}{i}.$$

In particular if $i = n$, then $\wedge^i V$ is a one dimensional vector space, with basis

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n,$$

if $v_1, v_2, \dots, v_n$ is a basis of V. So

$$\bigwedge^n \phi \colon \bigwedge^n V \longrightarrow \bigwedge^n V$$

is a map between one dimensional vector spaces. Now any such map is determined by a scalar. Indeed if W is a one-dimensional vector space and w is any non-zero vector in W, and ψ is any linear map, then

$$\psi(w) = aw,$$

and it is easy to see that a is independent of w.

Definition 4.7. *Let V be a vector space of dimension n and let ϕ be a linear map. Let $a \in F$ be the unique scalar such that $\bigwedge^n \phi$ is simply multiplication by a. Then a is called the* ***determinant*** *of ϕ and is denoted* $\det \phi$.

Example 4.8. *Let V be a two dimensional vector space and let ϕ be a linear map. Let us compute the determinant in this case.*

Pick a basis v and w for V. Then a basis of $\bigwedge^2 V$ is $v \wedge w$. Now expand ϕ in terms of this basis.

Suppose that

$$\phi(v) = av + bw$$

and

$$\phi(w) = cv + dw.$$

In other words suppose that the matrix of ϕ in the basis v and w is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\begin{aligned}\wedge^2\phi(v\wedge w) &= (av+bw)\wedge(cv+dw)\\ &= av\wedge(cv+dw)+bw\wedge(cv+dw)\\ &= ac(v\wedge v)+ad(v\wedge w)+bc(w\wedge v)+bd(w\wedge w)\\ &= ad(v\wedge w)-bc(v\wedge w)\\ &= (ad-bc)(v\wedge w)\\ &= \det(\phi)(v\wedge w).\end{aligned}$$

Lemma 4.9. *Let M, N and P be three R-modules and let $\phi\colon M\longrightarrow N$ and $\psi\colon N\longrightarrow P$ be two R-linear maps*

Then $\wedge^n(\phi\circ\psi)=\wedge^n(\phi)\circ\wedge^n(\psi)$.

Proof. Easy. □

In fact, this just says we have a functor.

Proposition 4.10. *Let V be a vector space of dimension n and let ϕ and ψ be two linear maps.*

Then $\det(\psi\circ\phi)=\det(\psi)\det(\phi)$.

Proof. Easy consequence of (4.9) and the definition of the determinant. □

Lemma 4.11. *Let V be a vector space over a field F, of characteristic not equal to two.*

Then there is a canonical isomorphism

$$V\otimes V\simeq \operatorname{Sym}^2 V\oplus\bigwedge^2 V.$$

Proof. Let U be the vector subspace of $V\otimes V$ generated by the vectors

$$v\otimes w+w\otimes v.$$

I claim that U is isomorphic to $\operatorname{Sym}^2 V$. Indeed define a map $V\times V$ to U by sending (v,w) to $v\otimes w+w\otimes v$. It is easy to see that this map is symmetric bilinear. Thus there is an induced R-linear map

$$\operatorname{Sym}^2 V\longrightarrow U.$$

It is easy to see that this map is both surjective. Thus it is an isomorphism, since both sides have the same dimension.

Similarly let W be the vector subspace generated by elements of the form

$$v\otimes w-w\otimes v.$$

It is easy to show that this is isomorphic to $\bigwedge^2 V$ (identifying $v\wedge w$ with $v\otimes w-w\otimes v$.

On the other hand, W and U span the whole of $V \otimes V$. We have

$$2(v \otimes w) = (v \otimes w + w \otimes v) + (v \otimes w - w \otimes v).$$

So that $2(v \otimes w)$ is in the span of U and W. As 2 is invertible, it follows that $v \otimes w$ is in the span, so that U and W span $V \otimes V$. As they have complimentary dimension, the result follows. $\square$