2. Noetherian Rings and Modules

First we need some more notation. We want to talk about both rings and modules being finitely generated. Therefore we need to introduce some notation for the subring generated by a set, distinguishable from the module (i.e ideal) generated by the same set.

Definition 2.1. *Let R be a ring, let S be a subring and let X be a subset of R. The smallest ring generated by $X \cup S$ is denoted $S[X]$.*
We say that X generates R, over S, if $S[X] = R$.

We have already seen some examples of this. For example, the Gaussian integers are denoted $\mathbb{Z}[i]$. One can think of this as the smallest ring in $\mathbb{C}$ containing $\mathbb{Z}$ and i.

As already observed, if R is a ring, then R need not even be finitely generated as a ring, but as an R-module, R is generated by 1, $R = \langle 1 \rangle$.

Definition 2.2. *Let M be an R-module.*
*We say that M is **Noetherian** if every submodule is finitely generated. We say that a ring is **Noetherian** if it is Noetherian as a module over itself.*

Clearly every PID is Noetherian, since in a PID, every ideal has one generator. In particular every field is Noetherian and moreover every Euclidean domain is Noetherian, so that the polynomial ring over a field is Noetherian, and both $\mathbb{Z}$ and $\mathbb{Z}[i]$ are Noetherian.

Note that a vector space is Noetherian if and only if it has finite dimension.

Definition 2.3. *Let X be a set. A **partial order** on X is a reflexive and transitive relation on $X \times X$. It is customary to denote a partial order $\leq$. The fact that $\leq$ is reflexive is equivalent to $x \leq x$ and the fact the $\leq$ is transitive is equivalent to*

$$a \leq b \qquad and \qquad b \leq c \qquad implies \qquad a \leq c.$$

We also require that if $x \leq y$ and $y \leq x$ then $x = y$.
We say that X satisfies the ascending chain condition (ACC) if every infinite increasing chain

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots$$

eventually stabilises, that is, there is an n_0 such that $x_n = x_m$ for every n and m at least n_0.

Note that, in the definition of a partial order, we do not require that every two elements of X are comparable. In fact if every pair of

elements are comparable, that is, for every x and $y \in X$, either $x \leq y$ of $y \leq x$, then we say that our partial order is a *total order*.

There is a similar notion for descending chains, knows as the descending chain condition, or DCC for short.

Example 2.4. *Every finite set with a partial order satisfies ACC and DCC for obvious reasons.*

Let X be a subset of the real numbers with the obvious relation. Then X is a partially ordered set (totally ordered, even). The set

$$X = \{\,\frac{1}{n} \,|\, n \in \mathbb{N}\,\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\},$$

satisfies ACC but it clearly does not satisfy DCC.

Let Y be a set and let X be a subset of the power set of Y, so that X is a collection of subsets of Y. Define a relation $\leq$ by the rule,

$$A \leq B \qquad \text{if and only if} \qquad A \subset B.$$

In the case that X is the whole power set of Y, note that $\leq$ is not a total order, provided that Y has at least two elements a and b, since in this case $A = \{a\}$ and $B = \{b\}$ are incomparable.

Lemma 2.5. *An R-module M is Noetherian if and only if the set of submodules of M satisfies ACC.*

Proof. Suppose that M is Noetherian and let

$$N_1 \subset N_2 \subset N_3 \ldots$$

be an ascending chain of submodules of M. Let N be the union. It is easy to check that N is a submodule of M. As M is Noetherian, N is finitely generated. Suppose that $n_1, n_2, \ldots, n_k$ are generators of N. For each $1 \leq i \leq k$, there is an index a_i, such that $n_i \in N_{a_i}$. Let a be the maximum of the a_i. Then $n_i \in N_j$, for all $1 \leq i \leq k$. It follows that

$$N = \langle n_1, n_2 \ldots, n_k \rangle \subset N_a \subset N_l \subset N,$$

for all $l \geq a$. But then we must have equality, so that $N = N_l$, for all $l \geq j$. Thus the set of submodules of M satisfies ACC.

Now suppose that the set of submodules of M satisfies ACC. Suppose that M is not Noetherian. Let N be a submodule of M which is not finitely generated. Pick a sequence of elements of N recursively, as follows. Put $n_0 = 0$. If we have already picked $n_1, n_2, \ldots, n_k$, then let N_k be the submodule they generate. Clearly $N_k \subset N$. N_k must be a proper subset of N, as we are assuming that N is not finitely generated. Pick $n_{k+1} \in N - N_k$. Then

$$N_1 \subset N_2 \subset N_3 \ldots$$

is a strictly ascending chain of submodules, a contradiction.   □

Definition 2.6. *Suppose that we have a sequence of R-modules,*

$$\ldots M_{i-1} \xrightarrow{f} M_i \xrightarrow{g} M_{i+1} \ldots$$

*We say that this sequence is **exact at** M_i, if the kernel of g is equal to the image of f. We say that this sequence is **exact**, if it is exact at each term. A **short exact sequence** is an exact sequence*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$$

Lemma 2.7.   *(1)*

$$0 \longrightarrow M \xrightarrow{f} N$$

   is exact at M if and only if f is injective.
   (2)

$$M \xrightarrow{f} N \longrightarrow 0$$

   is exact at N if and only if f is surjective.

Proof. Easy.   □

Note that there is an obvious short exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0.$$

Note also the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

where the first map is multiplication by 2 (note that this map is a module homomorphism but not a ring homomorphism).

Lemma 2.8. *Let*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

be a short exact sequence of R-modules.
   Then N is Noetherian if and only if M and P are Noetherian.

Proof. Suppose that N is Noetherian. Then M is (isomorphic to) a submodule of N. Thus any submodule of M is automatically a submodule of N and hence automatically finitely generated. Thus M is certainly Noetherian. There are two ways to see that P is Noetherian.

First suppose we are given a submodule P' of P. Let N' be the inverse image of P'. Then N' is a submodule of N. As N is Noetherian, it follows that N' has a finite number of generators. The image of these generators in P, generate the image of N', which as $N \longrightarrow P$ is surjective, is equal to P'. Thus P' is finitely generated.

Here is another way to prove that P is Noetherian. Pick an ascending chain of submodules
$$P_1 \subset P_2 \subset P_3 \dots$$
of P. Let
$$N_1 \subset N_2 \subset N_3 \dots$$
be the corresponding ascending chain in N. As N is Noetherian this chain must stabilise. Thus there is an index k, such that $N_i = N_j$, for all i and j at least k. As $N \longrightarrow P$ is surjective, it follows that the image of N_i is P_i. In particular $P_i = P_j$ for all i and j at least k. Thus every ascending chain of modules in P stabilises and P is Noetherian.

Now suppose that M and P are Noetherian. Suppose that
$$N_1 \subset N_2 \subset N_3 \dots$$
is an ascending chain in N. Let P_i be the image of N_i and let M_i be the intersection of N_i with M. Then we get two ascending chains, one in M,
$$M_1 \subset M_2 \subset M_3 \dots,$$
and one in P,
$$P_1 \subset P_2 \subset P_3 \dots.$$
As M and P are Noetherian, both chains eventually stabilise. It follows that we can find a common index k, so that $M_i = M_j$ and $P_i = P_j$, for all i and j at least k. On the other hand, note that if $M_i = M_j$ and $P_i = P_j$, then in fact $N_i = N_j$. Thus every chain of submodules in N eventually stabilises. $\square$

Proposition 2.9. *Every finitely generated module M over a Noetherian ring R is Noetherian.*

Proof. By induction on n and the short exact sequence
$$0 \longrightarrow R^{n-1} \longrightarrow R^n \longrightarrow R \longrightarrow 0,$$
we see that the direct sum of R with itself n times is Noetherian. On the other hand we have already seen that every finitely generated module is a quotient of a R^n, for some n. Thus M is a quotient of a Noetherian module and so it is Noetherian. $\square$

Theorem 2.10 (Hilbert's Basis Theorem)**.** *Let R be a Noetherian ring. Then $R[x]$ is Noetherian.*

Proof. Let I be an ideal of $R[x]$. We have to find a finite set of generators of I.

Let $J \subset R$ be the set of all coefficients of the leading terms of every element of I. I claim that J is an ideal in R. Suppose that a and b belong to J. Then there are two polynomials $f(x)$ and $g(x) \in I$ such

that the leading coefficient of $f(x)$ is a and the leading coefficient of $g(x)$ is b,
$$f(x) = ax^n + \ldots \qquad g(x) = bx^m + \ldots.$$
We may as well assume that $m \leq n$. Multiplying g by x^{n-m}, we may then assume that $m = n$. In this case the leading coefficient of $f + g$ is $a + b$. Now suppose that $r \in R$. Then $rf \in I$ has leading coefficient ra, so that $ra \in J$. Thus J is an ideal.

As R is Noetherian, J has a finite set of generators, $a_1, a_2, \ldots, a_k$. Pick $f_i(x) \in I$ such that the leading coefficient of $f_i(x)$ is a_i. Let m be the maximum of the degrees d_i of each $f_i(x)$.

Pick a polynomial $f(x)$ in I. Suppose that the degree d of $f(x)$ is at least m. Let a be the leading coefficient of $f(x)$. As $a \in J$, we may find $r_1, r_2, \ldots, r_k$ such that
$$a = r_1 a_1 + r_2 a_2 + \cdots + r_k a_k.$$
Consider
$$g(x) = f(x) - \sum r_i x^{d-d_i} f_i(x).$$
Then the coefficient of x^d in $g(x)$ is zero, so that $g(x)$ has degree less than d. Continuing in this way, we can find $h(x) \in \langle f_1, f_2, \ldots, f_k \rangle$, so that
$$f(x) = g(x) + h(x),$$
where $g(x)$ has degree at most $m-1$. Let M be the R-module generated by $1, x, x^2, \ldots x^{m-1}$. Then $g(x) \in I \cap M$. As R is Noetherian and M is finitely generated, M is Noetherian. As $I \cap M$ is a submodule of M, it follows that we may find a finite set of generators $g_1, g_2, \ldots, g_l$ for $I \cap M$. In this case $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_l$ are a finite set of generators for I.

Thus $R[x]$ is Noetherian. $\qquad\square$