EIGHTH HOMEWORK, DUE MONDAY MARCH 13TH

1. Let K be a field and let $K(x)$ be the field of rational functions in the indeterminate x. If $t \in K(x)$ but not in K then find a formula for the degree of the extension

$$[K(x) : K(t)].$$

2. Let K be any field, not of characteristic two, and let $L = K(t)$ be the field of rational functions, where t is an indeterminate. Let G be the group generated by the automorphisms $t \longrightarrow 1 - t$ and $t \longrightarrow 1/t$. Show that G is isomorphic to S_3 and identify the fixed field of G.
3. Given a finite group G, show that there is a field extension L/K with Galois group G.
4. Let L/K be a Galois field extension of degree a power of 2, and suppose that the characteristic of K is not equal to 2. Show that we may find a tower of intermediary subfields M_i/M_{i-1}, such that M_i is generated by an element α_i, where $\alpha_i^2 \in M_{i-1}$.
5. Find the Galois groups of the following extensions of $\mathbb{Q}$:
(a) $\mathbb{Q}(\sqrt{2}, \sqrt{5})$
(b) $\mathbb{Q}(\omega)$, where is a primitive cube root of one.
6. Find the Galois groups of (the splitting fields) of the polynomial $x^4 - 3x^2 + 4$ over the indicated fields
(a) $\mathbb{Q}$.
(b) $\mathbb{F}_2$.
(c) $\mathbb{F}_3$.
(d) $\mathbb{F}_4$.
7. For parts (5) and (6) find the subgroups of the Galois group, the intermediary fields, the normal subgroups and normal intermediary extensions and match them up.

Challenge Problems: 8. Let K be any field, not of characteristic two or three, containing an element i, such that $i^2 = -1$, and let $L = K(t)$ be the field of rational functions, where t is an indeterminate. Let G be the set of the following automorphisms, which fix K and send to t to

$$\pm t, \pm 1/t, \pm\frac{i(t+1)}{(t-1)}, \pm\frac{i(t-1)}{(t+1)}, \pm\frac{i(t+i)}{(t-i)}, \pm\frac{i(t-i)}{(t+i)}.$$

Show that G is a group isomorphic to the group of rotations of a regular tetrahedron and find the fixed field.

9. Let $p_1, p_2, \ldots, p_n$ be a collection of distinct primes. Let $K = \mathbb{Q}$, and let L be the field extension obtained by adjoining the square roots of every prime belonging to the list. Show that $[L : K] = 2^n$.