FOURTH HOMEWORK, DUE FEBRUARY 8TH

1. Let M, N and P be R-modules over a ring R. Show that there are natural isomorphisms:
(i)

$$\bigwedge^{d}(M \oplus N) \simeq \bigoplus_{i+j=d} \left(\bigwedge^{i} M \underset{R}{\otimes} \bigwedge^{j} N \right).$$

(ii)

$$\operatorname{Hom}_R(M \underset{R}{\otimes} N, P) \simeq \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, P)).$$

2. Let V and W be vector spaces over a field F. Let

$$V^* = \operatorname{Hom}_F(V, F),$$

be the dual vector space. Show that there is a natural isomorphism

$$\operatorname{Hom}^f_F(V, W) \simeq V^* \underset{F}{\otimes} W,$$

where

$$\operatorname{Hom}^f_F(V, W) \subset \operatorname{Hom}_F(V, W),$$

is the subset of linear maps whose image is finite dimensional.
3. Suppose that

$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

is a sequence of R-modules.
Show that

$$0 \longrightarrow \operatorname{Hom}_R(P, Q) \longrightarrow \operatorname{Hom}_R(N, Q) \longrightarrow \operatorname{Hom}_R(M, Q),$$

is (left) exact for all R-modules Q if and only if the first sequence is (right) exact.
4. Suppose that

$$0 \longrightarrow M \longrightarrow N \longrightarrow P,$$

is a sequence of R-modules.
Show that

$$0 \longrightarrow \operatorname{Hom}_R(Q, M) \longrightarrow \operatorname{Hom}_R(Q, N) \longrightarrow \operatorname{Hom}_R(Q, P),$$

is left exact for all R-modules Q if and only if the first sequence is left exact.
5. Suppose that

$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

is a right exact sequence of R-modules.

Show that

$$M \underset{R}{\otimes} Q \longrightarrow N \underset{R}{\otimes} Q \longrightarrow P \underset{R}{\otimes} Q \longrightarrow 0.$$

is right exact for all R-modules Q.
6. Give examples to show that one cannot extend (3–5) to short exact sequences. For example, even if

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

is a short exact sequence of R-modules then

$$0 \longrightarrow \operatorname{Hom}_R(M, Q) \longrightarrow \operatorname{Hom}_R(N, Q) \longrightarrow \operatorname{Hom}_R(M, Q) \longrightarrow 0,$$

is not necessarily a short exact sequence.
Challenge Problem: 7. Give an example of a PID R and a matrix A with entries in R such that one cannot realise the gcd d of A as an entry of A by elementary row and column operations.