PRACTICE PROBLEMS FOR THE SECOND MIDTERM

1. Give the definition of:
   (i) the factorial of an integer.
   (ii) the Fibonacci sequence.
   (iii) the Golden ratio.
   (iv) equality of two sets.
   (v) a subset.
   (vi) the emptyset.
   (vii) union; intersection; difference; the symmetric difference.
   (viii) the power set.
   (ix) monotonic increasing sequence; monotonic decreasing sequence; monotonic sequence.
   (x) upper bound of a set; lower bound of a set.
   (xi) infimum of a set; supremum of a set.
   (xii) function.
   (xiii) composition of functions.
   (xiv) identity function.
   (xv) injective; surjective; bijective;
   (xvi) inverse function.
2. Find the powerset of
$$\{\,1, 2, \{\,1\,\}, \{\,1, 2\,\}\,\}.$$
3. Let A, B and C be three sets. If
$$A \cap B \subset A \cap C \qquad \text{and} \qquad A \cup B \subset A \cup C$$
then prove that $B \subset C$.
4. Let A, B and C be three sets. Prove that
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$
5. Prove or disprove:
$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ \forall z \in \mathbb{R},\ x + y = z.$$
6. Prove or disprove:
(a)
$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ -x^4 < y.$$
(b)
$$\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ -x^4 < y.$$

(c)

$$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ -x^3 < y.$$

(d)

$$\exists y \in \mathbb{R},\ \forall x \in \mathbb{R},\ -x^3 < y.$$

7. Let $A \subset \mathbb{Z}$. Translate

"A has a maximum"

into a statement that uses only symbols and quantifiers. Negate the statement. Find an example of a set A where the statement is true and another set where the statement is false.

8. Let $f\colon A \longrightarrow B$ be a function. Prove that

(a) f is injective if and only if either A is the emptyset or there is a function $g\colon B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A\colon A \longrightarrow A$.

(b) f is surjective if and only if there is a function $g\colon B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B\colon B \longrightarrow B$.