MODEL ANSWERS TO THE NINTH HOMEWORK

1. (a) As A and B have the same cardinality there is a bijection $f\colon A \longrightarrow B$.
Define a function

$$F\colon \wp(A) \longrightarrow \wp(B),$$

by sending the element $X \in \wp(A)$ to

$$F(X) = \{\, f(x) \,|\, x \in X \,\}.$$

As f is a bijection, it has an inverse g. Using g we may write down a function

$$G\colon \wp(B) \longrightarrow \wp(A),$$

by sending the element $Y \in \wp(B)$ to

$$G(Y) = \{\, g(y) \,|\, y \in Y \,\}.$$

We check that G is the inverse of F. We have to check that $G \circ F = \mathrm{id}_{\wp(A)}$ and $F \circ G = \mathrm{id}_{\wp(B)}$.
We first check that $G \circ F = \mathrm{id}_{\wp(A)}$. Both sides are functions from $\wp(A)$ to itself. We have

$$\begin{aligned}
(G \circ F)(X) &= G(F(X)) \\
&= G(\{\, f(x) \,|\, x \in X \,\}) \\
&= (\{\, g(f(x)) \,|\, x \in X \,\}) \\
&= (\{\, (g \circ f)(x)) \,|\, x \in X \,\}) \\
&= (\{\, x \,|\, x \in X \,\}) \\
&= X \\
&= \mathrm{id}_{\wp(A)}(X).
\end{aligned}$$

Thus $G \circ F = \mathrm{id}_{\wp(A)}$. $F \circ G = \mathrm{id}_{\wp(B)}$ by symmetry. Thus F and G are inverses of each other. Hence F is a bijection and so $\wp(A)$ has the same cardinality as $\wp(B)$.
(b) As A_1 and A_2 have the same cardinality there is a bijection $f_1\colon A_1 \longrightarrow A_2$. As B_1 and B_2 have the same cardinality there is a bijection $g_1\colon B_1 \longrightarrow B_2$. Let $X = A_1 \cup B_1$ and $Y = A_2 \cup B_2$. Define a function

$$h_1\colon X \longrightarrow Y$$

by the rule

$$h_1(x) = \begin{cases} f_1(x) & \text{if } x \in A_1 \\ g_1(x) & \text{if } x \in B_1. \end{cases}$$

Note that x does not belong to both A_1 and B_1 and so h_1 is a well-defined function.

As f_1 and g_1 are bijections they have inverses $f_2\colon A_2 \longrightarrow A_1$ and $g_2\colon B_2 \longrightarrow B_1$. Define a function

$$h_2\colon Y \longrightarrow X$$

by the rule

$$h_2(x) = \begin{cases} f_2(x) & \text{if } x \in A_2 \\ g_2(x) & \text{if } x \in B_2. \end{cases}$$

Note that x does not belong to both A_2 and B_2 and so h_2 is a well-defined function.

We check that h_2 is the inverse of h_1. We check that $h_2 \circ h_1 = \mathrm{id}_X$. Both sides are functions from X to X. If $x \in X$ then there are two cases. If $x \in A_1$ then $h_1(x) = f_1(x) \in A_2$. Thus

$$\begin{aligned}(h_2 \circ h_1)(x) &= h_2(h_1(x)) \\ &= h_2(f_1(x)) \\ &= f_2(f_1(x)) \\ &= (f_2 \circ f_1)(x) \\ &= \mathrm{id}_X(x).\end{aligned}$$

We can handle the case $x \in B_1$ by symmetry. Thus $h_2 \circ h_1 = \mathrm{id}_X$. It follows that $h_1 \circ h_2 = \mathrm{id}_Y$ by symmetry. Thus h_2 is the inverse of h_1. It follows that $A_1 \cup B_1$ has the same cardinality as $A_2 \cup B_2$.

(c) As A_1 and A_2 have the same cardinality there is a bijection $f_1\colon A_1 \longrightarrow A_2$. As B_1 and B_2 have the same cardinality there is a bijection $g_1\colon B_1 \longrightarrow B_2$. Define a function

$$h_1\colon A_1 \times B_1 \longrightarrow A_2 \times B_2,$$

by the rule

$$h_1(a_1, b_1) = (f_1(a_1), g_1(b_1)).$$

As f_1 and g_1 are bijections they have inverses $f_2\colon A_2 \longrightarrow A_1$ and $g_2\colon B_2 \longrightarrow B_1$. Define a function

$$h_2\colon A_2 \times B_2 \longrightarrow A_1 \times B_1,$$

by the rule

$$h_2(a_2, b_2) = (f_2(a_2), g_2(b_2)).$$

We check that h_2 is the inverse of h_1. We first check that $h_2 \circ h_1 = \mathrm{id}_{A_1 \times B_1}$. We have

$$\begin{aligned}
(h_2 \circ h_1)(a_1, b_1) &= h_2(h_1(a_1, b_1)) \\
&= h_2(f_1(a_1), g_1(b_1)) \\
&= (f_2(f_1(a_1)), g_2(g_1(b_1))) \\
&= ((f_2 \circ f_1)(a_1), (g_2 \circ g_1)(b_1)) \\
&= (a_1, b_1) \\
&= \mathrm{id}_{A_1 \times B_1}(a_1, b_1).
\end{aligned}$$

Thus $h_2 \circ h_1 = \mathrm{id}_{A_1 \times B_1}$. By symmetry $h_1 \circ h_2 = \mathrm{id}_{A_2 \times B_2}$.
Thus h_2 is the inverse of h_1 and so h_1 is a bijection. But then $A_1 \times B_1$ has the same cardinality as $A_2 \times B_2$.
2. (a) We have already proved that $n \leq 2^n$ (besides, it is also a trivial case of Cantor's theorem), so that $n < 2^{n+1}$. Thus the set

$$A = \{\, m \in \mathbb{N} \,|\, n < 2^{m+1} \,\}$$

is non-empty. Let m be the smallest element of this set. Then $n < 2^{m+1}$ as $m \in A$.
We check that $2^m \leq n$. There are two cases. If $m = 0$ then $2^m = 1 \leq n$. Otherwise $m > 0$. As $m \in A$ is the smallest element, $m - 1 \notin A$. Thus $2^m = 2^{m-1+1} \leq n$. It follows that we may always find m such that $2^m \leq n < 2^{m+1}$. If l is another natural number with the same property then $l \in A$. $m \leq l$ by definition of m. If $l > m$ then $l \geq m + 1$ and so $n < 2^{m+1} \leq 2^l$, a contradiction. Thus $l = m$. This proves uniqueness.
(b) Let $P(n)$ be the statement that we may find a unique decreasing sequence of integers $m_1, m_2, \dots, m_k$ such that

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}.$$

Note that $P(1)$ holds since

$$1 = 2^0,$$

and this is the only way to write 1 as a sum of powers of 2.
Now suppose that $P(l)$ holds for all $l \leq n$. By (a) we may find a unique integer $m = m_1$ such that

$$2^{m_1} \leq n + 1 < 2^{m_1+1}.$$

Let

$$l = n + 1 - 2^{m_1}.$$

If $l = 0$ then we are done. Otherwise l is a natural number such that $0 < l \leq n$. Therefore $P(l)$ holds and we may find unique distinct natural numbers such that

$$l = 2^{m_2} + \cdots + 2^{m_k}$$

It follows that

$$n+1 = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}.$$

By assumption $m_i > m_{i+1}$ for $i > 1$. Note that

$$2^{m_2} \leq n+1.$$

Thus $m_2 \leq m_1$ by definition of m_1. Suppose that $m_1 = m_2$. Then

$$\begin{aligned} n+1 &\geq 2^{m_1} + 2^{m_1} \\ &= 2 \cdot 2^{m_1} \\ &= 2^{m_1+1}, \end{aligned}$$

contrary to our choice of m_1. Thus $m_1 > m_2$.
Suppose that there is another way to write $n+1$ as a sum of decreasing integers:

$$2^{p_1} + 2^{p_2} + \cdots + 2^{p_a} = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_b}.$$

Suppose that $p_1 < q_1$. As

$$2^{p_2} + \cdots + 2^{p_a} \leq 2^{p_2+1} - 1,$$

it follows that $p_2 \geq p_1$, a contradiction. Thus $p_1 = q_1$ by symmetry. But then

$$2^{p_2} + \cdots + 2^{p_a} = 2^{q_2} + \cdots + 2^{q_b}.$$

By induction $a = b$ and $p_i = q_i$, for all $2 \leq i \leq a$. Thus $P(n+1)$ holds.
(c) Define a function

$$f \colon \mathbb{N} \longrightarrow X,$$

as follows. If $n \in \mathbb{N}$ there are two cases: if $n = 0$ send this to the emptyset and if $n > 0$ send this to

$$A = \{ m_1, m_2, \dots, m_k \},$$

where

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k},$$

We already proved in (b) that this is a well-defined function.
Now define a function

$$g \colon X \longrightarrow \mathbb{N},$$

by sending the emptyset 0 and a non-empty set

$$A = \{ m_1, m_2, \dots, m_k \},$$

to

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k}.$$

It is clear that f and g are inverses of each other, from their definitions. Thus f is invertible, so that it is a bijection and so X is countable.
3. We first prove existence. We first do the case when $a > 0$ and $b \geq 0$. Let $P(b)$ be the statement that we may find integers q and r such that

$$b = qa + r.$$

We prove that $P(b)$ holds for all natural numbers by strong induction on b.
If $0 \leq b \leq a - 1$ then take $q = 0$ and $r = b$. Thus we know that $P(b)$ holds for any integer at least zero and no more than $a - 1$.
Now suppose that $P(b)$ holds for all $b \leq c$, where $c \geq a$. Consider $c+1$. Then

$$\begin{aligned} b &= c + 1 - a \\ &\leq c. \end{aligned}$$

By the inductive hypothesis we may find q and r such that

$$b = qa + r$$

and $0 \leq r < a$. In this case

$$\begin{aligned} c + 1 &= b + a \\ &= qa + r + a \\ &= (q + 1)a + r. \end{aligned}$$

Thus $P(c + 1)$ holds.
Thus by mathematical induction, $P(b)$ holds for all natural numbers. This proves existence of q and r, when $a > 0$ and $b \geq 0$.
Suppose that $b < 0$. Then $-b > 0$ and we may find q and r such that

$$-b = qa + r,$$

where $0 \leq r < a$.
There are two cases. If $r = 0$ then $b = qa$ and so $-b = (-q)a$. Thus

$$b = (-q)a + 0.$$

If $r > 0$ then

$$\begin{aligned} b &= -(qa + r) \\ &= -qa - r \\ &= -(q + 1)a + (a - r) \\ &= pa + s, \end{aligned}$$

where $p = -(q + 1)$ and $s = a - r$. Note that $s > 0$ as $r < a$. On the other hand, $s < a$ as $r > 0$. Thus we have existence if $a > 0$.

Suppose that $a < 0$. Then $-a > 0$ and so we may find q and r such that

$$-b = q(-a) + r,$$

where $0 < r < -a = |a|$. There are two cases. If $r = 0$ then $b = (-q)a$ and so

$$b = (-q)a + 0.$$

If $r > 0$ then

$$\begin{aligned} b &= -(q(-a) + r) \\ &= qa - r \\ &= (q+1)a - a - r \\ &= pa + s, \end{aligned}$$

where $p = q - 1$ and $s = -a - r$. As $r < -a$, $s > 0$. As $r > 0$, $s < -a = |a|$. Thus we have proved existence.

Now suppose that we may find q_1, r_1, q_2 and r_2, such that

$$q_1 a + r_1 = b = q_2 a + r_2.$$

Rearranging, we have

$$(q_1 - q_2)a = (r_1 - r_2).$$

Looking at the LHS, we have an integer divisible by a.

Note that $r_1 - r_2 < |a|$ as $r_1 < |a|$ and $r_2 > 0$. Note that $r_1 - r_2 > -a$ as $r_1 > 0$ and $-r_2 > -|a|$. Thus

$$-|a| < r_1 - r_2 < |a|.$$

Note the only integer divisible by a greater than $-|a|$ and less than $|a|$ is zero. Thus both sides of the equation are zero. As the RHS is zero, we have $r_1 - r_2 = 0$ so that $r_1 = r_2$. As the LHS is zero we have $(q_1 - q_2)a = 0$ and so $q_1 = q_2$ as a is non-zero.

This proves uniqueness.

4. The correct formula is:

$$|S| = \begin{cases} n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \cdots + (-1)^{n-1}\binom{n}{n-1} & \text{if } m \geq n \\ 0 & \text{if } m < n. \end{cases}$$

If $m < n$ then there are no surjective functions from A to B and so $|S| = 0$.

Thus we may suppose that $m \geq n$. We prove this formula by inclusion-exclusion. The first term n^m is simply the number of functions from A to B. So we just need to count the number of functions that are not surjective.

If a function is not surjective it misses one of the elements $b_1, b_2, \ldots, b_n$ of B. The number of functions which miss the element b_i is $(n-1)^m$, the number of functions from A to a set with $n-1$ elements. There are n possible choices of elements to miss. This is the second term, which needs to be excluded.

The problem is that we excluded too many functions. The functions which avoid two elements of B get excluded twice. If a function misses both b_i and b_j it got excluded twice, once as a function which misses b_i and once as a function which misses b_j. So we need to include these functions once. There are

$$\binom{n}{2}$$

choices of i and j and there are $(n-2)^m$ functions for each such choice. This is the third term which gets included.

The general term is where we miss k values of B. There are

$$\binom{n}{k}$$

choices of the elements of B we miss, there are $(n-k)^m$ functions for each such choice and a factor of $(-1)^{k-1}$ keeps track of whether we are excluding or including this term.