MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. Let $P(n)$ be the statement that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

We prove $P(n)$ holds for all natural numbers n by induction on n.
If $n=0$ the LHS is

$$(x+y)^0 = 1,$$

and the RHS is

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} &= \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k} \\ &= \binom{0}{0} x^0 y^0 \\ &= 1.\end{aligned}$$

As we have equality, $P(0)$ holds.
Now suppose that $P(m)$ holds. We check that $P(m+1)$ holds. We have

$$\begin{aligned}(x+y)^{m+1} &= (x+y)(x+y)^m \\ &= (x+y)\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \\ &= x\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} + y\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \\ &= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} \\ &= \sum_{k=1}^{m+1} \binom{m}{k-1} x^k y^{m-k+1} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} \\ &= y^{m+1} + \sum_{k=1}^{m} \left(\binom{m}{k-1} + \binom{m}{k} \right) x^k y^{m-k+1} + x^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m-k+1},\end{aligned}$$

where we use the inductive hypothesis to get from line one to line two and the fact that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

to get from line seven to line eight. Thus $P(k+1)$ holds.
Thus $P(n)$ holds for all natural numbers n, by mathematical induction, that is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2. Suppose that the elements of A are $a_1, a_2, \dots, a_m$ and the elements of B are $b_1, b_2, \dots, b_n$.
(a) A function $f\colon A \longrightarrow B$ is specified by choosing the images of the elements of A. For each $1 \leq i \leq m$ there are n choices for $f(a_i)$, one of $b_1, b_2, \dots, b_n$. Thus there are n^m functions from A to B,

$$|B^A| = n^m = |B|^{|A|}.$$

(b) There are two cases. If $m > n$ then there are no injective functions from A to B by the pigeonhole principle, so that $I = \emptyset$ and $|I| = 0$. On the other hand

$$|I| = n(n-1)\dots(n-m+1),$$

is zero, as one of the factors is zero. As both sides are zero, we have equality when $m > n$.
It remains to deal with interesting case when $m \leq n$. An injective function $f\colon A \longrightarrow B$ is specified by choosing the images of the elements of A. Now we pick the elements one at a time. There are n choices for $f(a_1)$. Having chosen the image of a_1 there are then $n-1$ choices for $f(a_2)$, we must make sure we don't send a_2 to $f(a_1)$. Suppose we have chosen the image of $f(a_1)$, $f(a_2)$, $\dots$, $f(a_k)$, $1 \leq k \leq m-1$. There are $n-k$ choices for where to send $f(a_{k+1})$.
Thus, by an obvious induction, we have

$$|I| = n(n-1)\dots(n-m+1),$$

3. (a) It is enough to show that $(0,1)$ and (a,b) have the same cardinality. Let

$$f\colon (0,1) \longrightarrow (a,b)$$

be the function $f(x) = a + (b-a)x$. Note that if $x \geq 0$ then

$$\begin{aligned} a + (b-a)x &\geq a + (b-a)\cdot 0 \\ &= a. \end{aligned}$$

On the other hand, if $x \leq 1$ then

$$\begin{aligned} a + (b-a)x &\leq a + (b-a) \cdot 1 \\ &= b. \end{aligned}$$

Thus $f(x) \in (a,b)$ and so f is well-defined.
Let

$$g \colon (a,b) \longrightarrow (0,1)$$

be the function

$$g(x) = \frac{x-a}{b-a}.$$

Note that if $x \geq a$ then

$$\begin{aligned} \frac{x-a}{b-a} &\geq \frac{0}{b-a} \\ &= 0. \end{aligned}$$

On the other hand, if $x \leq b$ then

$$\begin{aligned} \frac{x-a}{b-a} &\leq \frac{b-a}{b-a} \\ &= 1. \end{aligned}$$

Thus $g(x) \in (0,1)$ and so g is well-defined.
On the other hand

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(a + (b-a)x) \\ &= \frac{a + (b-a)x - a}{b-a} \\ &= x \\ &= \mathrm{id}_{(0,1)}(x). \end{aligned}$$

Thus $g \circ f = \mathrm{id}_{(0,1)}$. Similarly

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\frac{x-a}{b-a}) \\ &= a + (b-a)\frac{x-a}{b-a}) \\ &= a + x - a \\ &= x \\ &= \mathrm{id}_{(a,b)}(x). \end{aligned}$$

Thus $f \circ g = \mathrm{id}_{(a,b)}$. It follows that g is the inverse of f so that f is a bijection. Thus $(0,1)$ and (a,b) have the same cardinality. By

symmetry $(0,1)$ and (c,d) have the same cardinality. Thus (a,b) and (c,d) have the same cardinality.
(b) Let

$$f\colon (0,1) \longrightarrow \mathbb{R}$$

be the map given by

$$f(x) = \frac{1}{1-x} - \frac{1}{x} = \frac{2x-1}{(1-x)x}.$$

First note that f is defined on the open interval. It is somewhat of a pain to check using elementary methods that f is a bijection (since we have to solve quadratic equations).
If one assumes the standard results of calculus it is very straightforward. First note that f is continuous with a continuous derivative. Further the derivative of f is always positive, so that f is monotonic increasing. It follows that f is injective. As x approaches zero, $f(x)$ approaches $-\infty$; as x approaches one, $f(x)$ approaches ∞. As f is continuous, it follows that f is surjective by the intermediate value theorem.
(c) We showed in (a) that (a,b) has the same cardinality as $(0,1)$ and we showed in (b) that $(0,1)$ has the same cardinality as $\mathbb{R}$. It follows that (a,b) has the same cardinality as $\mathbb{R}$.
4. (a) Not injective. In fact $(0,0)$ and $(2,3)$ are two different points of the domain with the same image 0.
This function is surjective. If $n \in \mathbb{Z}$ then

$$f(n,n) = 3n - 2n = n.$$

(b) This function is injective and surjective. In fact l is its own inverse,

$$\begin{aligned}
(l \circ l)(B) &= l(A \triangle B) \\
&= A \triangle (A \triangle B) \\
&= (A \triangle A) \triangle B \\
&= \emptyset \triangle B \\
&= B \\
&= \mathrm{id}_B(B),
\end{aligned}$$

where we used some basic properties of the symmetric difference we established in previous homeworks. As l is invertible, it is a bijection.

(c) If $Y = X$ then

$$\begin{aligned} r(B) &= X \cap B \\ &= B \\ &= \mathrm{id}_{\wp(X)}(B), \end{aligned}$$

so that $r = \mathrm{id}_{\wp(X)}$. In this case r is a bijection.
Now suppose that Y is a proper subset. Then r is not injective but it is surjective. Note that $r(\emptyset) = \emptyset \cap Y = \emptyset$. As Y is a proper subset it follows that we can find $x \in X \setminus Y$. Let $B = \{ x \} \in \wp(X)$, the subset of X with the single element x. Then $r(B) = \{ x \} \cap Y = \emptyset$. As $B \neq \emptyset$ it follows that r is not injective.
Now suppose that $C \in \wp(Y)$. Then $C \subset Y \subset X$. Thus $C \subset X$ and so $C \in \wp(X)$. $r(C) = C \cap Y = C$. Thus r is surjective.
5. (a) We proved in a previous homework that if $A \in \wp(I)$, that is, $A \subset I$, then $A \in X_E$ if and only if $A \triangle \{1\} \in X_O$. It follows that f and g are indeed well-defined functions.
(b) We check that g is the inverse of f. $g \circ f$ and id_{X_E} are both functions from X_E to itself. So to check that $g \circ f = \mathrm{id}_{X_E}$ we just have to check they have the same effect on $A \in X_E$. We have

$$\begin{aligned} (g \circ f)(A) &= g(f(A)) \\ &= g(A \triangle \{1\}) \\ &= (A \triangle \{1\}) \triangle \{1\} \\ &= A \triangle (\{1\} \triangle \{1\}) \\ &= A \triangle \emptyset \\ &= A. \end{aligned}$$

Thus $g \circ f = \mathrm{id}_{X_E}$. By symmetry $f \circ g = \mathrm{id}_{X_O}$. Thus f and g are inverses of each other and so X_E and X_O have the same cardinalit.
Challenge problems/Just for fun:
6. Show that

(a) We apply the binomial theorem to $x = y = 1$. Then

$$\begin{aligned}
2^n &= (1+1)^n \\
&= (x+y)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k}.
\end{aligned}$$

(b) We apply the binomial theorem to $x = -1$ and $y = 1$. Then

$$\begin{aligned}
0 &= 0^n \\
&= (-1+1)^n \\
&= (x+y)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k}.
\end{aligned}$$

7. Let X be any set. Show that X and 2^X never have the same cardinality.

This will be proved in class.