MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. Let X be a finite set, and let A, B and $A_1, A_2, \dots, A_n$ be subsets of X. Let $A^c = X \setminus A$ denote the complement.
(a)

$$\sum_{x \in X} \chi_A(x) = |A|.$$

(b) We proved in homework six, question 2 (c) that

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B.$$

But we also proved homework six, question 2 (a) that

$$\chi_{A\cap B} = \chi_A\chi_B.$$

Thus

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}.$$

It follows that for every $x \in X$

$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x).$$

Summing over $x \in X$ we get

$$\sum_{x\in X} \chi_{A\cup B}(x) = \sum_{x\in X} \chi_A(x) + \sum_{x\in X} \chi_B(x) - \sum_{x\in X} \chi_{A\cap B}(x).$$

Applying part (a) to each term we get

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(c) We show the LHS is a subset of the RHS and vice-versa. If $a \in A^c \cap B^c$ then $a \in A^c$ and $a \in B^c$. Then $a \notin A$ and $a \notin B$. It follows that $a \notin A \cup B$. Thus $a \in (A \cup B)^c$. Therefore the RHS is a subset of the LHS.
Now suppose that $a \in (A \cup B)^c$. Then $a \notin A \cup B$. Thus $a \notin A$ and $a \notin B$. But then $a \in A^c$ and $a \in B^c$. Thus $a \in A^c \cap B^c$. Therefore the LHS is a subset of the RHS.
As we have containment both ways, we have proved that

$$A^c \cap B^c = (A \cup B)^c.$$

(d) Let $P(n)$ be the statement that

$$\chi_{(\cup_{i=1}^n A_i)^c} = \prod_{i=1}^{n} (1 - \chi_{A_i}).$$

We prove $P(n)$ holds by induction on n.

If $n = 1$ then the LHS is

$$\begin{aligned}\chi_{(\cup_{i=1}^n A_i)^c} &= \chi_{A_1^c} \\ &= \chi_{X \setminus A_1} \\ &= 1 - \chi_{A_1}\end{aligned}$$

and the RHS

$$\prod_{i=1}^n (1 - \chi_{A_i}) = 1 - \chi_{A_1}.$$

As both sides are equal, $P(1)$ holds.
Now suppose that $P(k)$ holds. Let

$$A = \bigcup_{i=1}^k A_i \qquad \text{and} \qquad B = A_{k+1}.$$

Then

$$\begin{aligned}\chi_{(\cup_{i=1}^{k+1} A_i)^c} &= \chi_{A \cup B)^c} \\ &= \chi_{A^c \cap B^c} \\ &= \chi_{A^c}\chi_{B^c} \\ &= \left(\prod_{i=1}^k (1 - \chi_{A_i})\right)(1 - \chi_{A_{K+1}}) \\ &= \prod_{i=1}^{k+1} (1 - \chi_{A_i}),\end{aligned}$$

where we used part (c) to get from line one to line two, homework 6, question 2, part (a) to get from line two to line three and the induction hypothesis to get from line three to line four. Thus $P(k+1)$ holds.
As we have show that $P(1)$ holds and that $P(k) \implies P(k+1)$, by the principle of mathematical induction, it follows that $P(n)$ holds for all n, that is,

$$\chi_{(\cup_{i=1}^n A_i)^c} = \prod_{i=1}^n (1 - \chi_{A_i}).$$

(e) Let $P(n)$ be the statement that

$$\prod_{i=1}^n (1 - x_i) = \sum_{k=0}^n (-1)^k \sum_{1 \le i_1 < i_2 < i_3 < \cdots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

We prove $P(n)$ holds by induction on n.

If $n = 1$ then the LHS is

$$\prod_{i=1}^{n}(1 - x_i) = \prod_{i=1}^{n}(1 - x_i)$$
$$= (1 - x_1).$$

and the RHS is

$$\sum_{k=0}^{n}(-1)^k \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k} = \sum_{k=0}^{1}(-1)^k \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le 1} x_{i_1}x_{i_2}\dots x_{i_k}$$
$$= 1 - x_1.$$

As both sides are equal, $P(1)$ holds.
Now suppose that $P(m)$ holds (we don't use k simply because k appears in the formula). We have

$$\begin{aligned}
\prod_{i=1}^{m+1}(1 - x_i) &= (1 - x_{m+1})\prod_{i=1}^{m}(1 - x_i) \\
&= (1 - x_{m+1})\left(\sum_{k=0}^{m}(-1)^k \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le m} x_{i_1}x_{i_2}\dots x_{i_k}\right) \\
&= \sum_{k=0}^{m}(-1)^k \sum_{1\le i_1<\cdots<i_k\le m} x_{i_1}x_{i_2}\dots x_{i_k} - x_{m+1}\left(\sum_{k=0}^{m}(-1)^k \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}\right) \\
&= \sum_{k=0}^{m}(-1)^k \sum_{1\le i_1<\cdots<i_k\le m} x_{i_1}x_{i_2}\dots x_{i_k} + \sum_{k=0}^{m}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le m} x_{i_1}x_{i_2}\dots x_{i_k}x_{m+1} \\
&= \sum_{k=0}^{m}(-1)^k \sum_{1\le i_1<i_2<\cdots<i_k\le m} x_{i_1}x_{i_2}\dots x_{i_k} + \sum_{k=1}^{m+1}(-1)^k \sum_{1\le i_1<i_2<\cdots<i_k=m+1} x_{i_1}x_{i_2}\dots x_{i_k} \\
&= \sum_{k=0}^{m+1}(-1)^k \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le m+1} x_{i_1}x_{i_2}\dots x_{i_k},
\end{aligned}$$

where we used the inductive hypothesis to get from line one to line two.
By mathematical induction it follows that $P(n)$ holds for all n. It follows that

$$\begin{aligned}
\chi_{(\cup_{i=1}^{n}A_i)^c} &= \sum_{k=0}^{n}(-1)^k \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le n} \chi_{\cap_{j=1}^{k}A_{i_j}} \\
&= 1 - (\chi_{A_1} + \cdots + \chi_{A_n}) + (\chi_{A_1\cap A_2} + \cdots + \chi_{A_{n-1}\cap A_n}) + \cdots + (-1)^n\chi_{A_1\cap A_2\cdots\cap A_n},
\end{aligned}$$

substituting $x_i = \chi_{A_i}$ and using the fact that

$$\chi_{\cap A_i} = \prod \chi_{A_i}.$$

(f) If we sum over $x \in X$ we get

$$\sum_{x\in X}(\chi_{(\cup_{i=1}^n A_i)^c}) = \sum_{k=0}^{n}(-1)^k \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le n}\ \sum_{x\in X}\chi_{\cap_{j=1}^k A_{i_j}}.$$

It follows that

$$|(\cup_{i=1}^n A_i)^c| = \sum_{k=0}^{n}(-1)^k \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le n} |\cap_{j=1}^k A_{i_j}|.$$

As

$$|(\cup_{i=1}^n A_i)^c| = |X| - |\cup_{i=1}^n A_i|$$

and

$$\sum_{x\in X} 1 = |X|,$$

we conclude that

$$|X| - |\cup_{i=1}^n A_i| = |X| - \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le n} |\cap_{j=1}^k A_{i_j}|.$$

so that

$$|\bigcup_{i=1}^n A_i| = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<i_2<i_3<\cdots<i_k\le n} |\bigcap_{j=1}^k A_{i_j}|.$$

2. (a)

$|A\cup B\cup C| = |A|+|B|+|C|-|B\cap C|-|A\cap C|-|A\cap B|+|A\cap B\cap C|.$

(b) Let A be the integers between 1 and 1000 divisible by 2, B be the integers between 1 and 1000 divisible by 3 and C be the integers between 1 and 1000 divisible by 5, so that

$$\begin{aligned} A &= \{\, k \in \mathbb{Z} \,|\, 1 \le k \le 1000,\ k \text{ is divisible by } 2 \,\} \\ B &= \{\, k \in \mathbb{Z} \,|\, 1 \le k \le 1000,\ k \text{ is divisible by } 3 \,\} \\ C &= \{\, k \in \mathbb{Z} \,|\, 1 \le k \le 1000,\ k \text{ is divisible by } 5 \,\}. \end{aligned}$$

We use the formula in (a) to count the number of elements of $A\cup B\cup C$. These are the integers divisible by at least one of 2, 3, or 5. Suppose that $a \in A$. Then we can find $k \in \mathbb{Z}$ such that $a = 2k$. As $1 \le a \le 1000$, we have $1 \le k \le 500$. Thus

$$|A| = 500$$

Similarly,

$$|C| = 200.$$

Now $b \in B$ if and only if $b = 3k$, some integer k. As $1 \leq b \leq 1000$, $1 \leq k \leq 333$. Thus

$$|B| = 333.$$

Now

$$A \cap B = \{\, k \in \mathbb{Z} \,|\, 1 \leq k \leq 1000,\ k \text{ is divisible by } 6 \,\}.$$

Thus $a \in A \cap B$ if and only if $a = 6k$ for some integer k. We have $1 \leq k \leq 166$. Thus

$$|A \cap B| = 166.$$

Similarly

$$|A \cap C| = 100 \qquad \text{and} \qquad |B \cap C| = 66.$$

Finally,

$$A \cap B \cap C = \{\, k \in \mathbb{Z} \,|\, 1 \leq k \leq 1000,\ k \text{ is divisible by } 30 \,\}.$$

Thus $a \in A \cap B \cap C$ if and only if $a = 30k$ for some integer k. We have $1 \leq k \leq 33$. Thus

$$|A \cap B \cap C| = 33.$$

It follows that

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C| \\
&= 500 + 333 + 200 - 166 - 100 - 66 + 33 \\
&= 734.
\end{aligned}$$

Thus the number of integers between 1 and $1,000$ not divisible by one of 2, 3 or 5 is

$$266.$$

3. (a) Suppose that f is injective and A is not the emptyset. We have to show that there is a function $g\colon B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A\colon A \longrightarrow A$. Pick an element a_0 of A. Define a function

$$g\colon B \longrightarrow A$$

as follows. If $b \in B$ there are two cases. If there is an element $a \in A$ such that $f(a) = b$ then define $g(b) = a$. Note that a is unique by injectivity. If there is no such element then define $g(b) = a_0$.
We check that $g \circ f = \mathrm{id}_A$. As both sides of the equation are functions from A to A it suffices to check that they have the same effect on arbitrary element $a \in A$. Let $b = f(a)$. Then $g(b) = a$ by definition of g.

$$\begin{aligned}
(g \circ f)(a) &= g(f(a)) \\
&= a \\
&= \mathrm{id}_A(a),
\end{aligned}$$

where we used the observation above to get from line one to line two. We now prove the other direction. If A is the emptyset then there is only one function from A to B and this function is injective. Otherwise suppose that there is a function $g\colon B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A\colon A \longrightarrow A$. We check that f is injective.
Suppose that a_1 and $a_2 \in A$ and $f(a_1) = f(a_2)$. If we apply g to both sides we get

$$\begin{aligned} a_1 &= \mathrm{id}_A(a_1) \\ &= (g \circ f)(a_1) \\ &= g(f(a_1)) \\ &= g(f(a_2)) \\ &= (g \circ f)(a_2) \\ &= \mathrm{id}_A(a_2) \\ &= a_2. \end{aligned}$$

Thus f is injective.
(b) Suppose that f is surjective. Define a function $g\colon B \longrightarrow A$ as follows. If $b \in B$ then pick an element a of A such that $f(a) = b$. As f is surjective we may find at least one such element. We check that $f \circ g = \mathrm{id}_B\colon B \longrightarrow B$. As both sides are functions from B to B, we just have to check they have the same effect on an element b of B. We have

$$\begin{aligned} (f \circ g)(b) &= f(g(b)) \\ &= f(a) \\ &= b \\ &= \mathrm{id}_B(b), \end{aligned}$$

where we used the definition of g to get from line one to line two. Thus $f \circ g = \mathrm{id}_B$.
Now suppose that we may find a function $g\colon B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B\colon B \longrightarrow B$. Suppose that $b \in B$ and let $a = g(b)$. We have

$$\begin{aligned} f(a) &= f(g(b)) \\ &= (f \circ g)(b) \\ &= \mathrm{id}_B(b) \\ &= b. \end{aligned}$$

Thus f is surjective.
4. First suppose that B is the emptyset. Then A has to be the emptyset as there is a function from A to B. As A and C are in bijection, it

follows that C is the emptyset. In this case f and g are bijections and the result is clear.
By assumption $g \circ f \colon A \longrightarrow C$ is a bijection. It follows that there is a function $h \colon C \longrightarrow A$ such that $h \circ (g \circ f) = \mathrm{id}_A$ and $(g \circ f) \circ h = \mathrm{id}_C$. We have

$$\begin{aligned}\mathrm{id}_A &= h \circ (g \circ f) \\ &= (h \circ g) \circ f.\end{aligned}$$

It follows that f is injective by 3 (a). Similarly, We have

$$\begin{aligned}\mathrm{id}_C &= (g \circ f) \circ h \\ &= g \circ (f \circ h).\end{aligned}$$

By 3 (b) it follows that g is surjective.
Suppose that f is surjective. Then f is bijective. It follows that f is invertible. Let $k \colon B \longrightarrow A$ be the inverse of f. Then k is a bijection. As the composition of bijections is a bijection, we have $(g \circ f) \circ k$ is a bijection. On the other hand, we have

$$\begin{aligned}(g \circ f) \circ k &= g \circ (f \circ k) \\ &= g \circ \mathrm{id}_B \\ &= g.\end{aligned}$$

Thus g is a bijection. In particular it is injective.
Now suppose that g is injective. Then g is a bijection. Let $k \colon A \longrightarrow B$ be the inverse of g. Then k is a bijection. As the composition of bijections is a bijection, we have $k \circ (g \circ f)$ is a bijection. On the other hand, we have

$$\begin{aligned}k \circ (g \circ f) &= (k \circ g) \circ f \\ &= \mathrm{id}_A \circ f \\ &= f.\end{aligned}$$

Thus f is a bijection. In particular f is surjective.
Challenge problems/Just for fun:
5. (a) If G has zero or one vertices then the degree function is injective. So we may assume that G has at least two vertices and we have to show that the degree function is not injective. There are n numbers between 0 and $n-1$. But if one vertex has degree 0 there are no vertices of degree $n-1$ and if there is a vertex of degree $n-1$ there are no vertices of degree 0. Thus we must miss one of the n numbers from zero to $n-1$. As there n vertices and at most $n-1$ possible degrees, the degree sequence cannot be injective, by the pigeonehole principle.

(b) If $n = 0$ then G has no vertices and no edges. If $n = 1$ then G has one vertex and one edge. So we may assume that $n \geq 2$.
There are two cases. Suppose that there is a vertex of degree zero. Then there is no vertex of degree $n - 1$ and we must get all of the numbers from 0 to $n - 2$. If we remove the vertex of degree zero then we get a graph with $n - 1$ vertices and no vertex of degree zero.
If there is a vertex of degree $n - 1$ then consider the complement of G, the graph H you get by putting an edge in H is there is no edge in G. Then H has a vertex of degree zero, corresponding to the vertex of degree $n - 1$ in G and H has vertices of every degree up to $n - 2$. Thus there are two graphs where the degree function misses precisely one number from zero to $n - 1$.
One can construct the graphs recursively as follows. Start with the graph with two vertices and no edges. Take its complement to get the graph with two vertices and one edge. Now add a vertex of degree zero, to get a graph with three vertices. Take its complement to get another graph with three vertices and then add a vertex of degree zero to get a graph with four vertices, and so on.
Formally, let $G_1, G_2, \dots$ be the graphs with a vertex of degree zero and let $H_1, H_2, \dots$ be the graphs with a vertex of degree $n - 1$. G_2 is the graph with two vertices and no edges. If we have constructed G_n then H_n is the complement of G_n and G_{n+1} is constructed from H_n by adding a vertex of degree zero.