MODEL ANSWERS TO THE FIFTH HOMEWORK

1. (a)

$$\exists \epsilon > 0,\ \forall \delta > 0,\ |x-1| < \delta \wedge |x^2-1| > \epsilon.$$

(b)

$$\exists \epsilon > 0,\ \exists x \in \mathbb{R},\ \forall n \in \mathbb{Z},\ |x-n| > \epsilon.$$

(c)

$$\exists \epsilon > 0,\ \exists x \in \mathbb{R},\ \forall m, n \in \mathbb{Z},\ |x-m-n\alpha| > \epsilon.$$

2. (a) This is true and so we prove it. Let $x = -2018$. Then

$$\begin{aligned} y^2 &= |y|^2 \\ &\geq 0 \\ &> -1 \\ &= 2017 - 2018 \\ &= 2017 + x. \end{aligned}$$

(b) This is true and so we prove it. Let $x = y^3 - 2018$. Then

$$\begin{aligned} y^3 &= x + 2018 \\ &> x + 2017. \end{aligned}$$

(c) This is false and so we disprove it, that is, we prove the negation

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ y^3 < 2017 + x.$$

There are two cases. If $2017 + x \geq 0$ then let $y = -1$. Then

$$\begin{aligned} y^3 &= -1 \\ &< 0 \\ &\leq 2017 + x. \end{aligned}$$

Now suppose $2017 + x < 0$. Let $y = 2016 + x < 2017 + x$.

$$\begin{aligned} y &= 2016 + x \\ &= 2017 + x - 1 \\ &< -1. \end{aligned}$$

On the other hand,

$$|y| > |2017 + x| \geq 1$$

and so

$$\begin{aligned} y^2 &= |y|^2 \\ &> |y|(2017+x) \\ &> 2017+x. \end{aligned}$$

We have

$$\begin{aligned} y^3 &= y \cdot y^2 \\ &< -y^2 \\ &< 2017+x. \end{aligned}$$

(d) This is true and so we prove it. Pick an integer N such that

$$N > \frac{1000}{\epsilon}.$$

If $n > N$ then

$$\begin{aligned} \frac{1000}{n} &= \frac{1000}{n} \cdot 1 \\ &= \frac{1000}{n} \cdot \frac{\epsilon}{\epsilon} \\ &= \frac{\epsilon}{n} \cdot \frac{1000}{\epsilon} \\ &< \epsilon \cdot \frac{N}{n} \\ &< \epsilon \cdot \frac{n}{n} \\ &= \epsilon \cdot 1 \\ &= \epsilon. \end{aligned}$$

3. We prove the contrapositive, if $L_1 \neq L_2$ then we can find $\epsilon > 0$ such that $|L_1 - L_2| > \epsilon$.
As $L_1 \neq L_2$, either $L_1 > L_2$ or $L_1 < L_2$. Suppose that $L_1 > L_2$. Then $L_1 - L_2 > 0$. Let

$$\epsilon = \frac{L_1 - L_2}{2} > 0.$$

Then

$$\begin{aligned} |L_1 - L_2| &= L_1 - L_2 \\ &> \frac{L_1 - L_2}{2} \\ &= \epsilon. \end{aligned}$$

If $L_1 < L_2$ then $L_2 > L_1$ and by symmetry

$$|L_1 - L_2| > \epsilon = \frac{L_2 - L_1}{2}.$$

4.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{Z},\ (n \geq N) \implies |x_n - a| < \epsilon.$$

5. (a) We show that $a_n \geq \sqrt{2}$. It is easy to see that $a_n > 0$ and so it suffices to show $a_n^2 \geq 2$. $a_1 = 2 \geq \sqrt{2}$ (however, this won't be a proof by induction).

On the other hand,

$$\begin{aligned} a_{n+1}^2 - 2 &= \left(\frac{a_n}{2} + \frac{1}{a_n}\right)^2 - 2 \\ &= \left(\frac{a_n^2 + 2}{2a_n}\right)^2 - 2 \\ &= \frac{(a_n^2+2)^2}{4a_n^2} - 2 \\ &= \frac{a_n^4 + 4a_n^2 + 4 - 8a_n^2}{4a_n^2} \\ &= \frac{a_n^4 - 4a_n^2 + 4}{4a_n^2} \\ &= \frac{(a_n^2 - 2)^2}{4a_n^2} \\ &\geq 0. \end{aligned}$$

(b) We have

$$\begin{aligned} a_{n+1} &= \frac{a_n}{2} + \frac{1}{a_n} \\ &= \frac{a_n^2 + 2}{2a_n} \\ &< \frac{a_n^2 + a_n^2}{2a_n} \\ &= \frac{2a_n^2}{2a_n} \\ &= a_n, \end{aligned}$$

where we used (a) to get from line 2 to line 3.

Challenge problems/Just for fun:

(c) We have already seen that $\sqrt{2}$ is a lower bound. We just have to show there is no larger lower bound. Let α be the infimum, which we

know exists, as the reals are complete. Then $\alpha \geq \sqrt{2}$, as $\sqrt{2}$ is a lower bound.
Let

$$\beta = -\alpha + \sqrt{\alpha^2 - 2}.$$

Then β is a root of the quadratic equation

$$x^2 + (-2\alpha)x + 2 = 0,$$

so that

$$\beta^2 - 2\alpha\beta + 2 = 0.$$

Solving for α, it follows that

$$\alpha = \frac{\beta}{2} + \frac{1}{\beta}.$$

As $\alpha \geq \sqrt{2}$ it follows that $\beta \geq \alpha$. Suppose that $\beta > \alpha$. As α is the infimum of the sequence, and β is bigger than α, β is not a lower bound. It follows that we can find n such that

$$a_n < \beta.$$

In this case

$$\begin{aligned} \alpha &\leq a_{n+1} \\ &= \frac{a_n}{2} + \frac{1}{a_n} \\ &< \frac{\beta}{2} + \frac{1}{\beta} \\ &= \alpha, \end{aligned}$$

which is a contradiction. It follows that

$$\begin{aligned} \beta &= \alpha \\ &= \frac{\beta}{2} + \frac{1}{\beta}. \end{aligned}$$

But then

$$\alpha = \frac{\alpha}{2} + \frac{1}{\alpha}.$$

Solving for α we get

$$2\alpha^2 = \alpha^2 + 2,$$

so that

$$\alpha^2 = 2.$$

As $\alpha > 0$ we must have $\alpha = \sqrt{2}$.
6. (a)

$$\forall d \in \mathbb{Z},\ \exists k \in \mathbb{Z},\ (kd = n) \implies (d = 1 \vee (\exists l \in \mathbb{Z},\ 2l = d)).$$

If n is not a power of 2 then it has no odd divisor other than 1. Therefore, it suffices to write down the condition that every divisor d of n is either 1 or divisible by 2. d divides n if there is an integer k such that $kd = n$. d is divisible by 2 if there is an integer l such that $2l = d$.
(b)

$$\forall d \in \mathbb{Z},\ \exists k \in \mathbb{Z},\ (kd = n) \implies (d = 1 \vee (\exists l \in \mathbb{Z},\ 5l = d)).$$

Just replace 2 by 5 (and use the fact that 5 is prime).
(c) The predicate $P(n)$

$$\forall d \in \mathbb{Z},\ \exists k \in \mathbb{Z},\ (kd = n) \implies (d = 1 \vee (\exists l \in \mathbb{Z},\ (2l = d) \vee (5l = d)))$$

is only true if the only divisors d of n are 1, 2 and 5. This means $n = 2^a 5^b$. But how to impose the extra condition that $a = b$?
7. This is a very famous problem. To solve this problem, it helps to state this problem in terms of graph theory (and to follow what is going on it will help to draw some pictures). Imagine the people at the dinner party as vertices of a graph, with an edge between every pair of vertices (this is called a complete graph). Colour an edge red if the people know each other and blue if they don't. Three mutual acquaintances corresponds to three red edges and three mutual strangers corresponds to three blue edges.
So we have a red blue colouring of the edges of a complete graph with n vertices and we want to make sure there is either a red or a blue triangle. Clearly $n \geq 3$. If there are three vertices colour two edges red and one blue (or vice-versa). As there is no monochromatic triangle, $n = 3$ does not work. If $n = 4$ there are six edges. Pick a square and colour it blue and colour the other two edges red. As there is no monochromatic triangle, $n = 4$ does not work. If $n = 5$ then there are ten edges. Pick a pentagon and colour it blue; colour the remaining five edges red. Call the vertices a, b, c, d and e and suppose we have the obvious blue pentagon, ab, bc, cd, de and $ea = ae$. The other edges are ac, ce, eb, bd and da, a red pentagon. As there is no red or blue triangle, $n = 5$ does not work.
It turns out $n = 6$ works. Pick any vertex v. There are five edges with endpoint v coloured red or blue. Suppose at least two edges are coloured red. Then at least three edges are coloured blue. It follows that at least three edges are coloured red or at least three edges are coloured blue.
Let suppose at least three edges are coloured red. Suppose these edges are va, vb and vc. Consider the subgraph with vertices a, b and c. The three edges ab, ac and bc are coloured either red or blue. There are two cases. If all three edges are blue then a, b and c is a blue triangle.

Otherwise, one of the edges is red. Let's suppose it is the edge ab (after all, the names a, b and c of the vertices are arbitrary). Consider the three vertices v, a and b. Every edge is red and so we have a red triangle. Either way, we have a monochromatic triangle.
If at least three edges coming out of v are blue, by symmetry, we can again argue that there is a monochromatic triangle.
So the answer is we need to invite six people.
8. Define a sequence of real numbers $a_1, a_2, \dots$ by the rule

$$a_n = \begin{cases} 0 & \text{if } n = 1 \\ \sqrt{7 + a_{n-1}} & \text{if } n > 1 \end{cases}$$

One can check that a_n is monotonic increasing. If the supremum is α then one can check, as in question 5, that

$$\alpha = \sqrt{7 + \alpha}.$$

Thus

$$\alpha^2 = 7 + \alpha,$$

so that α is a root of the quadratic equation

$$x^2 - x - 7 = 0,$$

It follows that

$$\alpha = \frac{1 + \sqrt{29}}{2} \approx 3.1925824.$$

The answer is 3. Indeed,

$$
\begin{aligned}
3 &= \sqrt{9} \\
&= \sqrt{1 + 2 \cdot 4} \\
&= \sqrt{1 + 2 \cdot \sqrt{16}} \\
&= \sqrt{1 + 2 \cdot \sqrt{1 + 15}} \\
&= \sqrt{1 + 2 \cdot \sqrt{1 + 3 \cdot 5}} \\
&= \sqrt{1 + 2 \cdot \sqrt{1 + 3 \cdot \sqrt{25}}} \\
&= \sqrt{1 + 2 \cdot \sqrt{1 + 3 \cdot \sqrt{1 + 24}}} \\
&= \sqrt{1 + 2 \cdot \sqrt{1 + 3 \cdot \sqrt{1 + 4 \cdot 6}}} \\
&= \sqrt{1 + 2 \cdot \sqrt{1 + 3 \cdot \sqrt{1 + 4 \cdot \sqrt{36}}}} \\
&= \sqrt{1 + 2 \cdot \sqrt{1 + 3 \cdot \sqrt{1 + 4 \cdot \sqrt{1 + 35}}}},
\end{aligned}
$$

and so on.