MODEL ANSWERS TO THE SECOND HOMEWORK

1. (4.1) Suppose not, suppose that there are integers m and n such that

$$14m + 21n = 100.$$

We will derive a contradiction.
Note that

$$\begin{aligned} 14m + 21n &= 7 \cdot 2m + 7 \cdot 3n \\ &= 7(2m + 3n). \end{aligned}$$

As $2m + 3n$ is an integer, it follows that 7 divides the LHS of the equation

$$14m + 21n = 100.$$

It follows that 7 divides 100. Thus we may find an integer k such that $7k = 100$.
There are two cases. If $k \leq 14$ then

$$\begin{aligned} 7 \cdot k &\leq 7 \cdot 14 \\ &= 98. \end{aligned}$$

If $k \geq 15$ then

$$\begin{aligned} 7 \cdot k &\geq 7 \cdot 15 \\ &= 105. \end{aligned}$$

Since k is either at most 14 or at least 15 it follows that there is no integer k such that $7k = 100$, a contradiction. This contradiction arose from the assumption that there are integers m and n such that

$$14m + 21n = 100.$$

Thus there are no integer solutions to the equation

$$14m + 21n = 100.$$

(4.2) Suppose not, suppose that there is an integer such that n^2 is odd and n is not odd. We will derive a contradiction.
If n is not odd then it is even. If n is even then n^2 is even (see (3.2) of homework 1). This is a contradiction. Therefore n is odd if n^2 is odd.
(4.3) It suffices to prove the contrapositive of the statement if n^2 is odd then n is odd. This is the statement that if n is even then n^2 is even. But this is precisely (3.2) of homework 1. Therefore n is odd if n^2 is odd.

2. Since the product of an even integer with any other integer is even, it suffices to prove that either n is even or $n+1$ is even. If n is not even then n is odd and so there is an integer m such that $n = 2m+1$. In this case

$$\begin{aligned} n+1 &= (2m+1)+1 \\ &= 2m+2 \\ &= 2(m+1). \end{aligned}$$

As $m+1$ is an integer, it follows that $n+1$ is even.
Thus at least one of n and $n+1$ is even and so $n(n+1)$ is even.
3. By assumption we may find integers m_1 and n_1 such that $m = m_1 d$ and $n = n_1 d$. In this case

$$\begin{aligned} rm + sn &= rm_1 d + sn_1 d \\ &= (rm_1 + sn_1)d. \end{aligned}$$

As $rm_1 + sn_1$ is an integer, it follows that d divides $rm + sn$.
4. False. Take $a = 2$ and $b = 3$. Then a and b are integers and 6 does not divide either a or b, since 6 is bigger than both. But $ab = 6 = 6 \cdot 1$ is divisible by 6.
5. If d is a divisor of n then we can find an integer e such that $de = n$. If $d \leq \sqrt{n}$ then let $d' = d$. We have $1 < d' \leq \sqrt{n}$ and d' divides n.
Otherwise $d > \sqrt{n}$. Let $d' = e$. Then d' is a divisor of n. Suppose that $e > \sqrt{n}$. We will derive a contradiction. We have

$$\begin{aligned} n &= de \\ &> \sqrt{n}e \\ &> \sqrt{n}\sqrt{n} \\ &> n, \end{aligned}$$

a contradiction. Thus $e \leq \sqrt{n}$.
On the other hand, as d and n are positive, it follows that e is positive. We claim that $e \neq 1$. Suppose not. We will derive a contradiction. If $e = 1$ then

$$\begin{aligned} d &= d \cdot 1 \\ &= d \cdot e \\ &= n, \end{aligned}$$

a contradiction. Thus $e \neq 1$ and so $1 < e \leq \sqrt{n}$.
Either way, we may find a divisor d' of n such that $1 < d' \leq \sqrt{n}$.

6. The trick to proving this result is to go backwards on a scratch piece of paper. After you have figured out how to go backwards, then simply reverse the implications, and go forwards:
Note that, since $x-y$ is a real number and the square of a real number is non-negative, we have

$$(x-y)^2 \geq 0.$$

Expanding we get

$$x^2 - 2xy + y^2 \geq 0$$

so that adding $4xy$ to both sides

$$(x+y)^2 = x^2 + 2xy + y^2 \geq 4xy.$$

By assumption we have x and $x \geq 0$ so that $x+y \geq 0$ and $4xy \geq 0$. Now we'd like to take square roots. We have to check that this preserves the inequality:

Claim 0.1. *If a and b are non-negative real numbers and $a \geq b$ then $\sqrt{a} \geq \sqrt{b}$.*

Proof of (0.1). It suffices to prove the contrapositive, that if $\sqrt{a} < \sqrt{b}$ then $a < b$.
As $\sqrt{a} < \sqrt{b}$, we have

$$\begin{aligned} a &= \sqrt{a}\sqrt{a} \\ &< \sqrt{a}\sqrt{b} \\ &< \sqrt{b}\sqrt{b} \\ &= b. \end{aligned}$$

□

Note that $x+y$, $a = (x+y)^2$ and $b = 4xy$ are all positive. So if we take the square root of both sides we have

$$x + y \geq 2\sqrt{xy}.$$

Dividing both sides by xy we have

$$\frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}}.$$

It follows that

$$\sqrt{xy} \geq \frac{2}{\frac{1}{x} + \frac{1}{y}}.$$

7. Number the people 1 to 12 where number 1 is the person at the back and number 12 is the person at the front. Note that nobody is looking at player number 1, so there is no way player number 1 can make sure they call out the correct colour. It follows that the remaining

eleven players must call out their own colour. Of course player number i can can see the colours of players with higher numbers but not lower numbers, including their own. Now player number one can indicate to one of the other players their colour but how can player number two call our their own colour *and* also send information to the higher number players? The key to this question is to realise that the twelve people can send signals to each other using the order they call out their colour. In essence they can say three things, blue, red or they can keep quiet.

We now describe a winning strategy. First, here is what someone who is a spectator will see: The players will first call out all of the blue colours, in decreasing order starting from the blue player with the largest number and ending with the blue player with the smallest number, except possibly player number one. Then player number one will call out red and the remaining red players will call out red in increasing order.

Here is the strategy the players adopt to effect this. We describe the first couple of steps, to illustrate how things work. Then we describe the strategy each individual player follows. At the start player number one looks at all the other colours. If all of them are red, player number one calls out red. Then player number two knows there are no blue colours, so player number two calls out red, and so on, as described in the previous paragraph. At each stage player number i knows it is safe to call out red, since player number one has signalled that there are no blue colours.

Otherwise player number one sees a blue colour and so player number one says nothing. Now player number two looks to see if there are any blue colours. If everyone else is red, player number two knows they are blue and so player number two says blue. At this point the responsibility to speak goes back to player number one, who ignores player number two and now sees only red. Player number one now calls out red and the rest of the players call our red in increasing order.

Otherwise player number two sees a blue colour and so player number two says nothing. The responsibility to speak passes to player number three, who carries out the same strategy as player number two.

Finally we describe the strategy each person adopts. At each stage it is someone's turn to speak. Each player has to keep track of this. We assume that once you have said something you move to one side (if not, everyone has to remember who already spoke). Player number one is special and we leave them to last. If it is your turn to speak, you have three options. Call out red, call out blue, or say nothing. Call out red only if player number one has already called out red. Call blue if there are no blue colours in front of you. Otherwise don't say anything. It is your turn to speak if it was the turn of the player immediately behind you to speak and they opted to say nothing.

If you are player number one, you will never say blue. If it is your turn to speak, you will say red if there are only red colours in front of you. Otherwise you say nothing. It is your turn to speak right at the beginning and if someone just said blue.

8. We describe a winning strategy and at the same time we show that it works. After quite a bit of trial and error, one realises the following approach works.

A little bit of notation. Call a coin “good” if we know it is one of the eleven equal weight coins. We will refer to the coin whose weight is not equal to the other eleven coins the “bad” coin. At the beginning we don’t know any good coins, that is, we have no idea which coin is the bad one, nor if it is heavier of lighter.

First divide the coins into three groups of four. Compare two groups of four. There are two cases.

Case I: The two groups of four coins have equal weight.

In this case we know all eight coins are good coins and the bad coin is one of the four unweighed coins.

Pick three of the unweighed coins and weigh them against three good coins. Two things can happen.

Case I (a): The two groups of three coins have equal weight.

In this case we know the bad coin is the last unweighed coin. Compare it with any good coin to see if it is lighter or heavier than the rest.

Case I (b): The two groups of three coins have different weight.

In this case we know the bad coin is one of the three coins we just weighed. We also know if this coin is lighter or heavier than the other coins. The situation is symmetric, so let’s say the bad coin is heavier. Now pick two of these three coins and weigh them. If they are equal the other coin is the bad coin. If they are not equal the heavier coin is the bad coin.

Case II: The two groups of four coins have a different weight.

So now we know the bad coin is one of these eight coins and we know the other four coins are all good coins. Call the coins from the heavier pile, A, B, C and D. Call the lighter coins a, b, c and d.

Now compare A, a and a good coin with b, B and C. There are three possible outcomes.

Case II (a): A, a and the good coin have the same weight as b, B and C.

So now we know the bad coin is one of D, c, or d. Compare c and d. If they have equal weight D is the bad coin and it is heavier. If c and

d don't have equal weight, the bad coin is the lighter coin. It is lighter than every other coin.
Case II (b): A, a, and the good coin is heavier than b, B and C.
In this case, either A is the bad coin or b is the bad coin. Compare A with a good coin. If A is the same weight as a good coin then b is the bad coin and it is lighter. If A has a different weight to the good coin, it is the bad coin and it is heavier.
Case II (c): A, a, and the good coin is lighter than b, B and C.
In this case, one of a, B or C is the bad coin. Compare B and C. If they have equal weight, a is the bad coin and it is lighter than every other coin. If B and C have different weight then the heavier coin is the bad coin and it is heavier than every other coin.
Since we have exhausted every possibility and in every case we have determined the bad coin and whether it is lighter or heavier, this is a winning strategy.
9. Let $G = (V, E)$ be a graph. Let m be the number of edges. We claim that

$$\sum_{v \in V} d(v) = 2m.$$

If we grant the claim it is clear that the LHS is even, since the RHS is divisible by 2.
This result is sometimes called the hand-shaking lemma; if a group of people all meet and some of them shake hands then the total number of hands that get shaken is even, because every time there is a hand-shake, two people shake hands.
We now turn to a proof of the claim. Here is one way to see the equality. Every vertex in G is connected to an edge. If we enumerate all the edges this way, we get the sum on the LHS. But every edge $e = vw$ gets counted twice this way; once when we consider v and once when we consider w. Putting all of this together we get

$$\sum_{v \in V} d(v) = 2m.$$

Here is another way to see the claim. We create an auxiliary graph H. The vertex set of H is the union $V_1 \cup V_2$ of the vertex set of G and the edge set of G. The edges of H connect a vertex v of G to an edge e of G if and only if v is an endpoint of e in the graph G.
We count the number of edges in H. On the one hand, every edge in H has one endpoint in $V_1 = V$. The degree of the vertex v in H is the same as its degree in G. Thus the number of edges in H is equal to the

sum:

$$\sum_{v \in V} d(v).$$

On the other hand, one endpoint of every edge in H has an endpoint in V_2, which is an edge in G. This edge, as a vertex in H, is connected to two vertices in V_1, its two endpoints. Thus the number of edges in H is also

$$2m.$$

Therefore

$$\sum_{v \in V} d(v) = 2m.$$