MODEL ANSWERS TO THE FIRST HOMEWORK

1. (a) Let a be a real number. There are two cases. If $a \geq 0$ then

$$|a| = a \geq a.$$

If $a < 0$ then $0 > a$ and $-a > 0$, and so

$$\begin{aligned} |a| &= -a \\ &> 0 \\ &> a. \end{aligned}$$

Either way, $|a| \geq a$, for any real number a.

(b) Let b be a real number. There are two cases. If $b \geq 0$ then

$$|b| = b.$$

In this case, squaring both sides, we get

$$|b|^2 = b^2.$$

If $b < 0$ then

$$|b| = -b.$$

In this case, squaring both sides, we get

$$\begin{aligned} |b|^2 &= (-b)^2 \\ &= b^2. \end{aligned}$$

Either way, $|b|^2 = b^2$, for any real number b.

(c) We already know that

$$x \leq |x|,$$

for any real number x. Note that this implies

$$\begin{aligned} -x &\leq |-x| \\ &= |x|, \end{aligned}$$

for any real number x.

Also, note that if $p \leq r$ and $q \leq s$ then

$$\begin{aligned} p + q &\leq p + s \\ &\leq r + s. \end{aligned}$$

There are two cases. If $c + d \geq 0$ then

$$\begin{aligned} |c + d| &= c + d \\ &\leq |c| + |d|. \end{aligned}$$

If $c + d < 0$ then

$$\begin{aligned} |c+d| &= -(c+d) \\ &= -c + -d \\ &\leq |c| + |d|. \end{aligned}$$

Either way,

$$|c| + |d| \geq |c+d|.$$

2. (3.2) As a divides b we may find an integer k such that $b = ka$. As b divides c we may find an integer l such that $c = lb$.
In this case

$$\begin{aligned} c &= lb \\ &= l(ka) \\ &= (lk)a. \end{aligned}$$

Note that lk is an integer, as it is a product of the integers k and l. Thus a divides c.
(3.3) Suppose that a is an even integer. Then we may find an integer b such that $a = 2b$.
In this case

$$\begin{aligned} a^2 &= (2b)^2 \\ &= 4b^2 \\ &= 2(2b^2). \end{aligned}$$

As $2b^2$ is an integer, it follows that a^2 is even.
3. We start by proving the first statement:

$$d \mid ((a_1 + b_1) - (a_2 + b_2)).$$

We need a simple property of divisibility:
Suppose that d divides two integers a and b. Then we may find integers k and l such that $a = kd$ and $b = ld$. Therefore

$$\begin{aligned} a + b &= kd + ld \\ &= (k + l)d. \end{aligned}$$

As $k + l$ is an integer it follows that d divides $a + b$. In words d divides the sum of two integers divisible by d.
As

$$d \mid a = (a_1 - a_2) \qquad \text{and} \qquad d \mid b = (b_1 - b_2),$$

it follows that d divides

$$\begin{aligned} a + b &= (a_1 - a_2) + (b_1 - b_2) \\ &= (a_1 + b_1) - (a_2 + b_2). \end{aligned}$$

Now we turn to the second statement:

$$d \mid (a_1 b_1 - a_2 b_2).$$

We need another simple property of divisibility:
Suppose that d divides the integer a and b is any integer. Then we may find an integer k such that $a = kd$. Therefore

$$\begin{aligned} ab &= (kd)b \\ &= (kb)d. \end{aligned}$$

Thus d divides ab. In words, if d divides any multiple of an integer divisible by d.
As

$$d \mid a = (a_1 - a_2) \qquad \text{and} \qquad d \mid b = (b_1 - b_2),$$

we have

$$d \mid a_1 b = a_1(b_1 - b_2) \qquad \text{and} \qquad d \mid ab_2 = (a_1 - a_2)b_2.$$

Thus d divides the sum

$$\begin{aligned} a_1 b + ab_2 &= a_1(b_1 - b_2) + (a_1 - a_2)b_2 \\ &= a_1 b_1 - a_1 b_2 + (a_1 b_2 - a_2 b_2) \\ &= a_1 b_1 - a_2 b_2. \end{aligned}$$

4. "Which door would the other guard tell me leads to freedom?"
Label the doors A and B. Note the following. If you ask the guard who tells the truth which door leads to freedom he will point to door A. If you ask the guard who lies which door leads to freedom he will point to door B.
There are two cases. If you ask the guard who always tells the truth, he would tell you what the guard who always lies says. So he will tell you that the other guard would point to door B.
If you ask the guard who always lies, he would tell you the opposite of what the guard who always tells the truth says. So he will tell you that the other guard will point to door B.
Either way, you now know that door A leads to freedom.
5. The trick with this problem is to keep track of the number n of times I appears in the string. There are only two rules which ever change that number, rules (2) and (3). In rule (2), one doubles the number of I's, that is, n becomes $2n$. In rule (3) we remove three I's, that is n becomes $n - 3$.

At the beginning there is one I, that is, $n = 1$. If we ever arrive at MU then $n = 0$. So at the start n is not divisible by 3. If you double a number which is not divisible by 3, it still isn't divisible by 3. If you start with a number not divisible by 3 and you subtract 3, then it is not divisible by 3.
So whatever string you can create, it is certainly the case that n is not divisible by 3. So we can never create MU.