SECOND MIDTERM
MATH 109, UCSD, SPRING 17

You have 50 minutes.

There are 5 problems, and the total number of points is 60. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Section instructor:______________________________

Section Time:______________________________

Problem	Points	Score
1	15	
2	15	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	60	

1. (15pts)

(i) *Give the definition of the emptyset.*

If A is any set the emptyset is

$$\emptyset = \{ \, x \in A \, | \, x \neq x \, \}.$$

(ii) *Give the definition of the powerset.*

If A is a set the powerset of A is the set of all subsets of A.

(iii) *Give the definition of an injective function.*

A function $f\colon A \longrightarrow B$ is injective if whenever a_1 and $a_2 \in A$ and $f(a_1) = f(a_2)$ then $a_1 = a_2$.

2. (15pts) *Let A, B and C be three sets. Prove that*
(a) $A \triangle \emptyset = A$,

We have

$$\begin{aligned} A \triangle \emptyset &= A \cup \emptyset \setminus A \cap \emptyset \\ &= A \setminus \emptyset \\ &= A. \end{aligned}$$

(b) $A \triangle A = \emptyset$,

We have

$$\begin{aligned} A \triangle A &= A \cup A \setminus A \cap A \\ &= A \setminus A \\ &= \emptyset. \end{aligned}$$

(c) if $A \triangle B = A \triangle C$ then $B = C$.

There are two ways to prove this.
The first is to prove this element by element. We first show that $B \subset C$. Suppose that $b \in B$. We have to show that $b \in C$. There are two cases. Suppose that $b \notin A$. Then $b \in B \setminus A$ so that $b \in A \triangle B$. Thus $b \in A \triangle C$. As $b \notin A$, $b \notin A \setminus C$ so that $b \in C \setminus A$. It follows that $b \in C$.
Now suppose that $b \in A$. Then $b \notin A \setminus B$ and $b \notin B \setminus A$ so that $A \triangle B$. Therefore $b \notin A \triangle C$. In particular $b \notin A \setminus C$. As $b \in A$ it follows that $b \in C$.
Either way, $b \in C$ and so $B \subset C$. By symmetry $C \subset B$. It follows that $B = C$.

Aliter:
We have

$$\begin{aligned} B &= B \triangle \emptyset \\ &= B \triangle (A \triangle A) \\ &= (B \triangle A) \triangle A \\ &= (C \triangle A) \triangle A \\ &= C \triangle (A \triangle A) \\ &= C \triangle \emptyset \\ &= C, \end{aligned}$$

where we used the identity

$$(E \triangle F) \triangle G = E \triangle (F \triangle G),$$

to get from lines two to three and from lines four to five, we used (a) on line one and to get from line six to seven and we used (b) to get from line one to two and line five to line six.

3. (10pts) *Write down the negation of the following propositions:*
(a)

$$\forall \epsilon > 0,\ \exists \delta > 0,\ |x-1| < \delta \implies |x^2 - 1| < \epsilon.$$

$$\exists \epsilon > 0,\ \forall \delta > 0,\ |x-1| < \delta \wedge |x^2 - 1| \geq \epsilon.$$

(b)

$$\forall \epsilon > 0,\ \forall x \in \mathbb{R},\ \exists n \in \mathbb{Z},\ |x-n| < \epsilon.$$

$$\exists \epsilon > 0,\ \exists x \in \mathbb{R},\ \forall n \in \mathbb{Z},\ |x-n| \geq \epsilon.$$

4. (10pts) Let X be a set. If $A \subset X$ is a subset of X the **characteristic function** of A is the function

$$\chi_A\colon X \longrightarrow \{0, 1\} \qquad \text{given by} \qquad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let A and B be two subsets of X. Prove that

$$(\forall x \in X,\ \chi_A(x) \leq \chi_B(x)) \iff (A \subset B).$$

Suppose that

$$\forall x \in X,\ \chi_A(x) \leq \chi_B(x).$$

We show that $A \subset B$.
Pick $a \in A$. Then, by definition of χ_A, we have $\chi_A(a) = 1$. As

$$1 = \chi_A(a) \leq \chi_B(a),$$

and the only possible values of $\chi_B(a)$ are 0 or 1, we must have $\chi_B(a) = 1$. But then by definition of χ_B, we have $a \in B$. It follows that $A \subset B$.
Now suppose that $A \subset B$. We show that

$$\forall x \in X,\ \chi_A(x) \leq \chi_B(x).$$

Pick $x \in X$. There are two cases.
If $\chi_B(x) = 0$ then, by definition of χ_B, we have $x \notin B$. As $A \subset B$ it follows that $x \notin A$. But then by definition of χ_A, $\chi_A(x) = 0$. In particular

$$\begin{aligned} \chi_A(x) &= 0 \\ &= \chi_B(x). \end{aligned}$$

Thus $\chi_A(x) \leq \chi_B(x)$.
Now suppose that $\chi_B(x) = 1$. As $\chi_A(x)$ is either 0 or 1, we have

$$\begin{aligned} \chi_A(x) &\leq 1 \\ &= \chi_B(x). \end{aligned}$$

Either way, $\chi_A(x) \leq \chi_B(x)$.
Thus

$$\forall x \in X,\ \chi_A(x) \leq \chi_B(x).$$

5. (10pts) *Let $f\colon A \longrightarrow B$ be a function, where A is a non empty set. Prove that f is injective if and only if there is a function $g\colon B \longrightarrow A$ such that $g \circ f = id_A\colon A \longrightarrow A$.*

Suppose that f is injective. Pick an element a_0 of A. Define a function

$$g\colon B \longrightarrow A$$

as follows. If $b \in B$ there are two cases. If there is an element $a \in A$ such that $f(a) = b$ then define $g(b) = a$. Note that a is unique by injectivity. If there is no such element then define $g(b) = a_0$.
We check that $g \circ f = \mathrm{id}_A$. As both sides of the equation are functions from A to A it suffices to check that they have the same effect on arbitrary element $a \in A$. Let $b = f(a)$. Then $g(b) = a$ by definition of g.

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= a \\ &= \mathrm{id}_A(a),\end{aligned}$$

where we used the observation above to get from line one to line two.
Now suppose that there is a function $g\colon B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A\colon A \longrightarrow A$. We check that f is injective.
Suppose that a_1 and $a_2 \in A$ and $f(a_1) = f(a_2)$. If we apply g to both sides we get

$$\begin{aligned}a_1 &= \mathrm{id}_A(a_1) \\ &= (g \circ f)(a_1) \\ &= g(f(a_1)) \\ &= g(f(a_2)) \\ &= (g \circ f)(a_2) \\ &= \mathrm{id}_A(a_2) \\ &= a_2.\end{aligned}$$

Thus f is injective.

Bonus Challenge Problems

6. (10pts) 6. (10pts) *Guess the limit*

$$\sqrt{5+\sqrt{5+\sqrt{5+\sqrt{5+\dots}}}}.$$

Show that your guess is an upper bound for the sequence of real numbers $a_1, a_2, \dots$ *defined by the rule*

$$a_n = \begin{cases} 0 & \text{if } n = 1 \\ \sqrt{5 + a_{n-1}} & \text{if } n > 1. \end{cases}$$

First note that we can make an educated guess as to limit. If a is the limit then we should have

$$a = \sqrt{5 + a},$$

so that a is a root of the quadratic equation

$$x^2 - x - 5 = 0.$$

As a is positive, it follows that

$$a = \frac{1 + \sqrt{21}}{2}.$$

We now check that a is an upper bound. We prove this by induction on n. If $n = 1$ then

$$a_n = 0 \leq a.$$

Now suppose that $a_k \leq a$. It suffices to check that $a_{k+1}^2 \leq a$. We have

$$\begin{aligned} a^2 - a_{k+1}^2 &= \left(\frac{1+\sqrt{21}}{2}\right)^2 - 5 - a_k \\ &= \frac{1 + 2\sqrt{21} + 21}{4} - 5 - a_n \\ &= \frac{1+\sqrt{21}}{2} - a_n \\ &= a - a_n \\ &> 0. \end{aligned}$$

Thus $a_n < a$ by induction on n. Thus a is an upper bound.

7. (10pts) *Let X be a set. Show that $\wp(X)$ and $\{0,1\}^X$ have the same cardinality.*

We want to define a bijection

$$\Theta\colon \wp(X) \longrightarrow \{0,1\}^X.$$

We send an element of the powerset, that is, a subset A of X, to its characteristic function χ_A, so that

$$\Theta(A) = \chi_A.$$

We check that Θ is a bijection.
We first check that Θ is injective. Suppose that A and $B \in \wp(X)$ and $\Theta(A) = \Theta(B)$. Then A and B are two subsets of X and $\chi_A = \chi_B$. As $\chi_A \leq \chi_B$, we proved in 5 that $A \subset B$. Similarly, as $\chi_B \leq \chi_A$, we proved in 5 that $B \subset A$. As $A \subset B$ and $B \subset A$ it follows that $A = B$. Thus Θ is injective.
Now we check that Θ is surjective. Suppose that $f \in \{0,1\}^X$. Then f is a function from X to $\{0,1\}$. Define a subset A of X as follows

$$A = \{ x \in X \mid f(x) = 1 \}.$$

We check that $\chi_A = f$. Both sides are functions from X to $\{0,1\}$. We check they have the same effect on $x \in X$.
There are two cases. If $f(x) = 1$ then $x \in A$ by definition of A and so $\chi_A(x) = 1$. If $f(x) = 0$ then $x \notin A$ by definition of A and so $\chi_A(x) = 0$. Either way, $\chi_A(x) = f(x)$ and $f = \chi_A = \Theta(A)$. Thus Θ is surjective.
It follows that Θ is a bijection and so $\wp(X)$ and $\{0,1\}^X$ have the same cardinality.

8. (10pts) *The axiom of foundation states that for every set x we can find an element y of x such that $y \cap x = \emptyset$.*
Use the axiom of foundation to show that for every set z,

$$z \notin z.$$

Let

$$x = \{ z \}.$$

We apply the axiom of foundation to the set x. We obtain an element y of x which is disjoint from x. Now x has only one element, z, and so $y = z$.
We have

$$\emptyset = y \cap x = z \cap x.$$

It follows that every element of x is not an element of z. But z is an element of x and so z is not an element of z, that is, $z \notin z$.