FIRST MIDTERM
MATH 109, UCSD, SPRING 17

You have 50 minutes.

There are 5 problems, and the total number of points is 65. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Section instructor:______________________________

Section Time:______________________________

Problem	Points	Score
1	15	
2	10	
3	20	
4	10	
5	10	
6	10	
7	10	
Total	65	

1. (15pts) *Let a and b be two integers.*
(i) *Give the definition of a divides b.*

We say that the integer a **divides** the integer b if there is an integer k such that $b = ka$.

(ii) *Give the definition of a is even.*

The integer a is **even** if 2 divides a.

(iii) *Give the definition of a is odd.*

The integer a is **odd** if it is not even.

2. (10pts) *Prove that*

$$|a| \geq a,$$

for all real numbers a.

There are two cases.
If $a \geq 0$ then

$$\begin{aligned} |a| &= a \\ &\geq a. \end{aligned}$$

If $a < 0$ then $|a| = -a$. As $a < 0$, we have $0 > a$ and $-a > 0$. Thus

$$\begin{aligned} |a| &= -a \\ &> 0 \\ &> a. \end{aligned}$$

Since either $a \geq 0$ or $a < 0$ and in both cases $|a| \geq a$, it follows that $|a| \geq a$ for all real numbers a.

3. (20pts) *Let a, b and c be integers. Prove or disprove:*
(i) *If a divides b and b divides c then a divides c.*

This result is true and so we will prove it.
As a divides b there is an integer k such that $b = ka$. As b divides c there is an integer l such that $c = lb$. It follows that

$$\begin{aligned} c &= lb \\ &= l(ka) \\ &= (lk)a. \end{aligned}$$

As lk is integer it follows that a divides c.

(ii) *If* 10 *divides* ab *then either* 10 *divides* a *or* 10 *divides* b*.*

This is not true and so we will disprove this result.
It suffices to give a single example. Let $a = 2$ and $b = 5$. Then

$$\begin{aligned} ab &= 2 \cdot 5 \\ &= 10 \\ &= 10 \cdot 1. \end{aligned}$$

Thus 10 divides the product ab.
But $a = 2 < 10$ and $b = 5 < 10$ and so 10 does not divide either a or b.
Thus $a = 2$ and $b = 5$ is a counterexample to the statement:
If 10 divides ab then either 10 divides a or 10 divides b.

4. (10pts) *If x and y are non-negative real numbers then prove* that
$$\sqrt{xy} \leq \frac{x+y}{2}.$$

We first prove:

Claim 0.1. *If a and b are non-negative real numbers and $a \geq b$ then $\sqrt{a} \geq \sqrt{b}$.*

Proof of (0.1). It suffices to prove the contrapositive, that if $\sqrt{a} < \sqrt{b}$ then $a < b$.
As $\sqrt{a} < \sqrt{b}$, we have
$$\begin{aligned} a &= \sqrt{a}\sqrt{a} \\ &< \sqrt{a}\sqrt{b} \\ &< \sqrt{b}\sqrt{b} \\ &= b. \end{aligned}$$
□

As $x - y$ is a real number, we have
$$(x-y)^2 \geq 0.$$
As $(x-y)^2 = x^2 - 2xy + y^2$ it follows that
$$x^2 - 2xy + y^2 \geq 0.$$
Adding $4xy$ to both sides we get
$$x^2 + 2xy + y^2 \geq 4xy.$$
As $(x+y)^2 = x^2 + 2xy + y^2$ it follows that
$$(x+y)^2 \geq 4xy.$$
As x and y are non-negative, it follows that $x+y$, $a = (x+y)^2$ and $b = 4xy$ are all non-negative. It follows by (0.1) that
$$x + y \geq 2\sqrt{xy}.$$
Dividing both sides by 2 we get
$$\sqrt{xy} \leq \frac{x+y}{2},$$
for all real numbers x and y.

5. (10pts) *Prove that*

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

We have to prove that $P(1)$ holds and that $P(k) \implies P(k+1)$ for any positive integer k.
If $n = 1$ then the LHS is

$$1^2 = 1,$$

and the RHS is

$$\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)2+1}{6} = 1.$$

As both sides are equal, it follows that $P(1)$ is true.
Now suppose that k is a positive integer and $P(k)$ holds. We check that $P(k+1)$ holds.
We have

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= [1^2 + 2^2 + 3^2 + \cdots + k^2] + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6},
\end{aligned}$$

where we use the fact that $P(k)$ is true to get from the first line to the second line. It follows that $P(k+1)$ holds.
As we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$ for any positive integer k, it follows that $P(n)$ holds for all n, that is

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Bonus Challenge Problems

6. (10pts) *Prove that*

$$17n^3 + 103n$$

is divisible by 6 for all positive integers n.

We prove this result by mathematical induction. Let $P(n)$ be the statement that

$$17n^3 + 103n$$

is divisible by 6.

We have to prove that $P(1)$ holds and that $P(k) \implies P(k+1)$ for any positive integer k.

If $n = 1$ then

$$\begin{aligned} 17n^3 + 103n &= 17 + 103 \\ &= 120 \\ &= 6 \cdot 24. \end{aligned}$$

Thus 6 divides $17n^3 + 103n$ when $n = 1$ and it follows that $P(1)$ is true.

Now suppose that k is a positive integer and $P(k)$ holds. We check that $P(k+1)$ holds.

We have

$$\begin{aligned} 17(k+1)^3 + 103(k+1) &= 17(k^3 + 3k^2 + 3k + 1) + 103(k+1) \\ &= 17k^3 + 51k^2 + 51k + 17) + 103k + 103 \\ &= (17k^3 + 103k) + 17 \cdot 3 \cdot k(k+1) + 120. \end{aligned}$$

Now $17k^3 + 103k$ is divisible by 6, as $P(k)$ holds. As we have already seen, 120 is also divisible by 6. Note $17 \cdot 3$ is divisible by 3 and $k(k+1)$ is always even and so $17 \cdot 3k(k+1)$ is divisible by 6. Thus $17(k+1)^3 + 103(k+1)$ is divisible by 6 and so $P(k+1)$ holds.

As we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$ for any positive integer k, it follows that $P(n)$ holds for all n, that is

$$17n^3 + 103n$$

is divisible by 6.

7. (10pts) *If n is a positive integer then find a formula for*

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1)$$

and prove it is correct.

Since there are n terms and each term is at worse quadratic, we guess that the answer is a cubic polynomial. If $n = 0$ then the sum is zero, so we guess this cubic is divisible by n. So we guess a formula like so:

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1) = n(an^2 + bn + c).$$

If $n = 1$ then we are supposed to get 3 and so $a + b + c = 3$. If $n = 2$ then we get 18 and so $4a + 2b + c = 9$. Subtracting we get $3a + b = 6$. If $n = 3$ we get 53 and so $53 = 3(9a + 3b + c)$. Subtracting we get $22 = 3(4a + b)$. Subtracting again, we get $4 = 3a$, so that $a = 4/3$. In this case $b = 2$ and $c = -1/3$.
We guess that

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1) = \frac{n}{3}(4n^2 + 6n - 1).$$

for every integer n. We prove this by mathematical induction.
Let $P(n)$ be the statement that the formula holds. If $n = 1$ the LHS is 3 and the RHS is

$$\frac{n}{3}(4n^2 + 6n - 1) = \frac{1}{3}(4 + 6 - 1) = 3.$$

Thus $P(1)$ is true.
Now suppose $P(k)$ is true; we check $P(k+1)$ holds:

$$\begin{aligned} 1 \cdot 3 + \cdots + (2k+1)(2k+3) &= [1 \cdot 3 + 3 \cdot 5 + \cdots + (2k-1)(2k+1)] + (2k+1)(2k+3) \\ &= \frac{k}{3}(4k^2 + 6k - 1) + (2k+1)(2k+3) \\ &= \frac{1}{3}(4k^3 + 6k^2 - k + 12k^2 + 24k + 9) \\ &= \frac{1}{3}(4k^3 + 18k^2 + 23k + 9) \\ &= \frac{k+1}{3}(4k^2 + 14k + 9) \\ &= \frac{k+1}{3}(4(k+1)^2 + 6(k+1) - 1), \end{aligned}$$

where we used $P(k)$ to get from line one to line two. Thus $P(k+1)$ is true. Mathematical induction implies that

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1) = \frac{n}{3}(4n^2 + 6n - 1).$$

for every integer n.

Aliter: There is another way to proceed. Since

$$(2i-1)(2i+1) = 4i^2 - 1,$$

in fact we can rewrite the sum as

$$\begin{aligned}
\sum_{i=1}^{n}(2i-1)(2i+1) &= \sum_{i=1}^{n} 4i^2 - 1 \\
&= 4(\sum_{i=1}^{n} i^2) - \sum_{i=1}^{n} 1 \\
&= 4\left(\frac{n(n+1)(2n+1)}{6}\right) - n \\
&= 2\left(\frac{n(n+1)(2n+1)}{3}\right) - n \\
&= \frac{2n(n+1)(2n+1) - 3n}{3} \\
&= \frac{n(2(n+1)(2n+1) - 3)}{3} \\
&= \frac{n(2(2n^2+3n+1) - 3)}{3} \\
&= \frac{n(4n^2+6n-1)}{3},
\end{aligned}$$

where we used 5 to go from the second line to the third line.
Note we could have stopped at line four, but we continued on just to make sure we got the same answer as before.