9. More about induction

Here we collect some more sophisticated topics centred around induction. First of all, how to find a formula for the sum of the first n squares?

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = ?$$

By analogy with the other cases, we first guess that the sum is a polynomial in n. Now we have n terms and each term is at most n^2. Therefore the sum is at most n^3. So it looks as though we have a polynomial of degree at most 3 (on the other hand, half of the terms are at least $(n/2)^2 = n^2/4$ and so the sum is at least

$$n^2/4 \cdot n/2 = n^3/8,$$

so almost certainly the formula involves a cubic polynomial).

If we imagine plugging in $n = 0$ then there are no terms in the sum and so the LHS is zero. But then our polynomial of degree 3 is divisible by n. The general such polynomial is

$$n(an^2 + bn + c)$$

and it is our job to determine a, b and c. We plug in small values of n to determine a, b and c. If $n = 1$ the LHS is 1. Thus

$$a + b + c = 1.$$

If $n = 2$ the LHS is 5 and so

$$2(4a + 2b + c) = 5.$$

If we multiply the first equation by 2 and subtract we get:

$$2(3a + b) = 3.$$

If $n = 3$ the LHS is 14 and so

$$3(9a + 3b + c) = 14.$$

Multiplying the first equation by 3 and subtracting we get

$$3(8a + 2b) = 11.$$

If we take the other equation involving only a and b, multiply by 3 and subtract, we get

$$3(2a) = 2.$$

Therefore $a = 1/3$. It follows that $b = 1/2$ and so $c = 1/6$. We guess a formula of the form

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}.$$

It is a homework problem to prove this is correct.

Sometimes one can do a double induction:

Theorem 9.1. *For all non-negative integers m and n we have*

$$F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}.$$

Proof. Let $P(m,n)$ be the statement that

$$F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}.$$

We prove this by double (strong) induction on m and n.

We have to check three things. We have to check that $P(0,0)$, $P(1,0)$, $P(0,1)$ and $P(1,1)$ all hold and and that $P(i,j)$ for all $i \leq p$ and $j \leq q$ implies both $P(p+1,q)$ and $P(p,q+1)$.

We first check that $P(0,0)$, $P(1,0)$, $P(0,1)$ and $P(1,1)$ all hold.

When $m = n = 0$ the LHS of the equation is

$$F_{m+n+1} = F_{0+0+1} = F_1 = 1$$

and the RHS of the equation is

$$F_m F_n + F_{m+1} F_{n+1} = F_0 F_0 + F_1 F_1 = 0 + 1 = 1.$$

As both sides are equal, $P(0,0)$ holds.

When $m = 1$ and $n = 0$, the LHS of the equation is

$$F_{m+n+1} = F_{1+0+1} = F_2 = 1$$

and the RHS of the equation is

$$F_m F_n + F_{m+1} F_{n+1} = F_1 F_0 + F_2 F_1 = 0 + 1 = 1.$$

As both sides are equal, $P(1,0)$ holds. By symmetry, $P(0,1)$ also holds.

When $m = 1$ and $n = 1$, the LHS of the equation is

$$F_{m+n+1} = F_{1+1+1} = F_3 = 2,$$

and the RHS of the equation is

$$F_m F_n + F_{m+1} F_{n+1} = F_1 F_1 + F_2 F_2 = 1 + 1 = 2.$$

As both sides are equal, $P(1,1)$ holds.

Thus $P(0,0)$, $P(1,0)$, $P(0,1)$ and $P(1,1)$ all hold.

Now assume that $P(i,j)$ holds for all $i \leq p$ and $j \leq q$. Suppose that $p \geq 1$. Let us show that $P(p+1,q)$ holds. We have

$$\begin{aligned}
F_{p+q+2} &= F_{p+q} + F_{p+q+1} \\
&= F_{p-1}F_q + F_p F_{q+1} + F_p F_q + F_{p+1} F_{q+1} \\
&= F_{p-1}F_q + F_p F_q + F_p F_{q+1} + F_{p+1} F_{q+1} \\
&= (F_{p-1} + F_p) F_q + (F_p + F_{p+1}) F_{q+1} \\
&= F_{p+1} F_q + F_{p+2} F_{q+1},
\end{aligned}$$

where we used the recursive definition of the Fibonacci numbers for the first line, the inductive hypotheses $P(p-1, q)$ and $P(p, q)$ to get from the first line to the second line, and the recursive definition of the Fibonacci numbers to get from the fourth line to the fifth line.

Therefore $P(p+1, q)$ holds. We have shown that $P(i, j)$ for all $i \leq p$ and $j \leq q$ implies $P(p+1, q)$. By symmetry, it follows that we can also deduce $P(p, q+1)$ using the same hypotheses.

It follows by induction that $P(m, n)$ holds for all non-negative integers m and n, that is,

$$F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}. \qquad \square$$