8. Variations on a Theme

Here we present some variations on the method of induction.

First we note that it is not really necessary to start at $n = 1$. Sometimes it makes sense to start at zero and sometimes it makes sense to start at a larger value of n.

Lemma 8.1. *For every integer $n \geq 4$,*

$$2^n < n!.$$

Proof. Let $P(n)$ be the statement that

$$2^n < n!.$$

We prove that $P(n)$ holds for all integers $n \geq 4$. We proceed by mathematical induction.

We first check that $P(4)$ holds. If $n = 4$ then the LHS is

$$2^n = 2^4 = 2 \cdot 2^3,$$

and the RHS is

$$n! = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 2^3 \cdot 3.$$

As $2 < 3$ it follows that

$$\begin{aligned} 2^4 &= 2 \cdot 2^3 \\ &< 3 \cdot 2^3 \\ &= 4!. \end{aligned}$$

Thus $P(4)$ is true.

We now check that $P(k) \implies P(k+1)$, for every $k \geq 4$. Assume that $P(k)$ holds. We check that $P(k+1)$ holds. We have

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! \\ &\leq (k+1) \cdot k! \\ &= (k+1)!, \end{aligned}$$

where we used the inductive hypothesis to get from line one to line two and the fact that $k \geq 1$ to get from line two to line three. Thus $P(k+1)$ holds.

We checked that $P(4)$ holds and that $P(k) \implies P(k+1)$ holds and so by the principle of mathematical induction $P(n)$ holds for all $n \geq 4$, that is, for every positive integer n,

$$2^n \leq n!.$$ □

Another place that mathematical induction turns up is in definitions, although in this context it is often called recursion. Here are three well-known examples.

Definition 8.2. *Let n be a non-negative integer.*

*The **factorial** of n, denoted $n!$, is defined recursively by*

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n > 0. \end{cases}$$

Definition 8.3. *Let $a_1, a_2, \ldots$ be a sequence of real numbers. The **sum** of the first n terms, denoted*

$$\sum_{i=1}^{n} a_i,$$

is defined recursively by

$$\sum_{i=1}^{n} a_i = \begin{cases} a_1 & \text{if } n = 1 \\ \sum_{i=1}^{n-1} a_i + a_n & \text{if } n > 1. \end{cases}$$

Definition 8.4. *Let a be a real number and let n be a non-negative integer. The **product** of a with itself n times, denoted*

$$a^n$$

is defined recursively by

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ a \cdot a^{n-1} & \text{if } n > 0. \end{cases}$$

It is possible to define sequences of integers with quite subtle properties using quite simple definitions:

Definition 8.5. *The **Fibonacci sequence** is the sequence of non-negative integers $F_0, F_1, F_2, \ldots$ defined recursively by*

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-2} + F_{n-1} & \text{if } n > 1 \end{cases}$$

The first few terms are therefore

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

Somewhat surprisingly there is a closed form expression for the nth term of the Fibonacci sequence:

Theorem 8.6 (Binet Formula)**.** *If n is a non-negative integer then*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

The numbers α and β are the two roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

α is called the *Golden ratio*. If you have a rectangle with sides in proportion to the Golden ratio and you remove a square from one end of length the shortest side then you get another rectangle whose sides have ratio the Golden ratio.

Note that $-1 < \beta < 0$, so that $0 < |\beta| < 1$. Therefore $|\beta|^n$ approaches zero, as n goes to infinity. In particular, F_n is the closest integer to $\alpha^n/\sqrt{5}$ and the ratio F_n/F_{n-1} approaches the Golden ratio.

To prove (8.6) we will need strong mathematical induction:

Axiom 8.7 (Induction Principle (bis))**.** *Let $P(n)$ be a statement about the positive integers.*

Then $P(n)$ is true for all positive integers, provided:

(1) $P(1)$ is true.
(2) ($P(j)$ for every $1 \leq j \leq k$) implies $P(k+1)$.

In other words, to deduce $P(k+1)$, you are free not only to assume $P(k)$, but you may also assume

$$P(1), \qquad P(2), \qquad P(3), \qquad \dots \qquad P(k-1) \qquad \text{and} \qquad P(k).$$

Proof of (8.6)*.* Let $P(n)$ be the statement that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

We prove this using strong mathematical induction.

We need to check two things.

If $n = 0$ then the LHS is

$$F_n = F_0 = 0,$$

and the RHS is

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0.$$

Thus $P(0)$ is true.

If $n = 1$ then the LHS is

$$F_n = F_1 = 1,$$

and the RHS is

$$\begin{aligned}\frac{\alpha^n - \beta^n}{\sqrt{5}} &= \frac{\alpha - \beta}{\sqrt{5}} \\ &= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\ &= \frac{2\sqrt{5}}{2\sqrt{5}} \\ &= 1.\end{aligned}$$

Thus $P(1)$ is true.

Now we assume $P(j)$ holds for every $0 \leq j \leq k$. We also assume that $k \geq 1$. We have

$$\begin{aligned}F_{k+1} &= F_{k-1} + F_k \\ &= \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}} + \frac{\alpha^k - \beta^k}{\sqrt{5}} \\ &= \frac{\alpha^k + \alpha^{k-1} - (\beta^k + \beta^{k-1})}{\sqrt{5}} \\ &= \frac{\alpha^{k-1}(1+\alpha) - \beta^{k-1}(1+\beta)}{\sqrt{5}} \\ &= \frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}},\end{aligned}$$

where we used the recursive definition of F_k on the top line, the fact that $P(k-1)$ and $P(k)$ hold by strong induction, to get from the first line to the second line, and we used the identities

$$1 + \alpha = \alpha^2 \qquad \text{and} \qquad 1 + \beta = \beta^2,$$

which hold as α and β are roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

Thus $P(k+1)$ holds.

As we checked the hypothesis of strong mathematical induction, it follows that $P(n)$ holds for all non-negative integers n, that is,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}. \qquad \square$$