7. Induction

Let n be a non-negative integer. Compute the following sums:

(1)
$$1 + 2 + 3 + \cdots + (n-1) + n.$$
(2)
$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1).$$
(3)
$$2 + 4 + 6 + \cdots + (2n-2) + 2n.$$

The first sum was famously computed by Gauss. Reverse the sequence, to get
$$n + (n-1) + (n-2) + \cdots + 1.$$
Now add the two sequences together to get
$$(n+1) + (n+1) + (n+1) + \cdots + (n+1).$$
Since there are n terms the sum is
$$n(n+1).$$
But this is twice what we want it to be, so
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

One might also guess that the second sum is n^2, just by looking at a few examples.

Induction gives a method to prove this formula, assuming you already have the formula.

Axiom 7.1 (Induction Principle)**.** *Let $P(n)$ be a statement about the positive integers.*

Then $P(n)$ is true for all positive integers, provided:

(1) $P(1)$ is true.
(2) $P(k)$ implies $P(k+1)$, for every positive integer k.

Theorem 7.2. *If n is a non-negative integer then*
$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

Proof. Let $P(n)$ be the statement that
$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$
We want to know that $P(n)$ is true for all positive integers n. We proceed by mathematical induction. We have to check two things.

First we check that $P(1)$ is true. If $n = 1$ the LHS is equal to

$$1$$

and the RHS is equal to

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1.$$

Thus $P(1)$ is true.

Now we check that $P(k)$ implies $P(k+1)$, for every positive integer k. Let k be a positive integer. Assume that $P(k)$ holds, that is, assume that

$$1 + 2 + 3 + \cdots + (k-1) + k = \frac{k(k+1)}{2}.$$

We want to check that $P(k+1)$ holds. We have

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= [1 + 2 + 3 + \cdots + k] + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}, \end{aligned}$$

where we used the inductive hypothesis $P(k)$ to get from line one to line two. Therefore we have

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2},$$

so that $P(k+1)$ holds.

We checked that $P(1)$ holds and that $P(k) \implies P(k+1)$ holds and so by the principle of mathematical induction $P(n)$ holds for all n, that is for every positive integer n,

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

□

Lemma 7.3. *If n is a non-negative integer then*

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2.$$

Proof. Let $P(n)$ be the statement that

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2.$$

We want to know that $P(n)$ is true for all positive integers n. We proceed by mathematical induction. We have to check two things.

First we check that $P(1)$ is true. If $n = 1$ the LHS is equal to

$$1$$

and the RHS is equal to

$$n^2 = 1^2 = 1.$$

Thus $P(1)$ is true.

Now we check that $P(k)$ implies $P(k+1)$, for every positive integer k. Let k be a positive integer. Assume that $P(k)$ holds, that is, assume that

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

We want to check that $P(k+1)$ holds. We have

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k+1) &= [1 + 3 + 5 + \cdots + (2k-1)] + (2k+1) \\ &= k^2 + 2k + 1 \\ &= (k+1)^2. \end{aligned}$$

where we used the induction hypothesis $P(k)$ to get from line one to line two. Therefore we have

$$1 + 3 + 5 + \cdots + (2k+1) = (k+1)^2$$

so that $P(k+1)$ holds.

We checked that $P(1)$ holds and that $P(k) \implies P(k+1)$ holds and so by the principle of mathematical induction $P(n)$ holds for all n, that is, for every positive integer n,

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2. \qquad \square$$

Lemma 7.4. *For every positive integer n,*

$$5^{2n-1} + 2^{2n-1}$$

is divisible by 7.

Proof. Let $P(n)$ be the statement that

$$5^{2n-1} + 2^{2n-1}$$

is divisible by 7.

We want to know that $P(n)$ is true for all positive integers n. We proceed by mathematical induction. We have to check two things.

First we check that $P(1)$ is true. If $n = 1$ then

$$\begin{aligned} 5^{2n-1} + 2^{2n-1} &= 5^{2-1} + 2^{2-1} \\ &= 5 + 2 \\ &= 7, \end{aligned}$$

which is divisible by 7.

Now we check that $P(k)$ implies $P(k+1)$, for every positive integer k. Let k be a positive integer. Assume that $P(k)$ holds, that is, assume that

$$5^{2k-1} + 2^{2k-1}$$

is divisible by 7.

We want to check that $P(k+1)$ holds. We have

$$\begin{aligned} 5^{2k+1} + 2^{2k+1} &= 5^2 5^{2k-1} + 2^2 2^{2k-1} \\ &= 5^2 5^{2k-1} + 5^2 2^{2k-1} - 5^2 2^{2k-1} + 2^2 2^{2k-1} \\ &= 5^2(5^{2k-1} + 2^{2k-1}) + (2^2 - 5^2) 2^{2k-1} \\ &= 5^2(5^{2k-1} + 2^{2k-1}) - 21 \cdot 2^{2k-1}. \end{aligned}$$

By induction, we know that

$$5^{2k-1} + 2^{2k-1}$$

is divisible by 7. Thus

$$5^2(5^{2k-1} + 2^{2k-1})$$

is divisible by 7. On the other hand,

$$21 \cdot 2^{2k-1},$$

is also divisible by 7. Thus

$$5^{2k+1} + 2^{2k+1}$$

is divisible by 7. It follows that $P(k+1)$ holds.

We checked that $P(1)$ holds and that $P(k) \implies P(k+1)$ holds and so by the principle of mathematical induction $P(n)$ holds for all n, that is for every positive integer n,

$$5^{2n-1} + 2^{2n-1}$$

is divisible by 7. □

Lemma 7.5. *For every positive integer n,*

$$n < 2^n.$$

Proof. Let $P(n)$ be the statement that

$$n < 2^n.$$

We want to show that $P(n)$ holds for all positive integers n. We proceed by mathematical induction. We have to check two things.

First we check that $P(1)$ is true. If $n = 1$ then the LHS is 1 and the RHS is 2 and $P(1)$ is true as $1 < 2$.

Now we check that $P(k)$ implies $P(k+1)$, for every positive integer k. Let k be a positive integer. Assume that $P(k)$ holds, that is, assume that

$$k < 2^k.$$

We want to check that $P(k+1)$ holds. We have

$$\begin{aligned} k+1 &\leq k+k \\ &< 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1}, \end{aligned}$$

where we used the inductive hypothesis $P(k)$ to get from line one to line two. It follows that $P(k+1)$ holds.

We checked that $P(1)$ holds and that $P(k) \implies P(k+1)$ holds and so by the principle of mathematical induction $P(n)$ holds for all n, that is for every positive integer n,

$$n < 2^n. \qquad \square$$